TRANSIENT HEAT CONDUCTION IN A SPHERE

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Introduction

Solutions to transient conduction problems with convection boundary conditions involve infinite series that usually converge rapidly. Evaluation of these series requires the computation of eigenvalues from equations that can only be solved by trial and error. Because this process is rather tedious, several approximate methods have been developed. These methods are less tedious to apply than the exact solution, but they must be used with care.

This article makes a comparison of the exact solution to transient heat conduction in a sphere to three approximate methods: the lumped capacity method, the Heisler chart method, and the Heat Balance Integral method. Mathcad software was used with each of these methods.

Approximations to the Exact Solution

One-dimensional, unsteady heat conduction in a sphere is governed by the following partial differential equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

This equation is usually solved by assuming that the temperature can be represented as the product of two functions, one a function of radius only and the other a function of time only. Application of boundary and initial conditions leads to the following general solution:

$$\frac{T(X,\tau)-T_{fluid}}{T(X,0)-T_{fluid}}=\sum_{i=1}^{\infty}A_i e^{-\lambda^2\tau}\frac{\sin(\lambda_i X)}{\lambda_i X}\quad[1]$$

where

$$\tau=\frac{\alpha t}{R^2}\ \text{ and }\ X=\frac{r}{R}$$

$$A_i=\frac{4\left(\sin(\lambda_i)-\lambda_i\cos(\lambda_i)\right)}{2\lambda_i-\sin(2\lambda_i)}\quad[2]$$

and λ_i *satisfies the eigenvalue equation*

$$1-\lambda_i\cot(\lambda_i)=Biot\ Number\quad[3]$$

$\lambda_1 := 1$ *Guess value*

Given

$$1-\lambda_1\cdot\cot(\lambda_1)=Bi$$

$\lambda_1 := Find(\lambda_1) = 1.543$

Figure 1
Finding Eigenvalues

The principal source of tedium in generating the exact solution is finding the eigenvalues λ_i, a process that can only be done by trial and error. Use of modern computer tools such as Mathcad can remove much of the tedium. Figure 1 shows part of a Mathcad worksheet that finds the first eigenvalue from Equation 3 above. Because trial and error is required, a Mathcad solve block is used. Additional eigenvalues were found by choosing other guess values. The values of A_1, A_2, and A_3 were calculated by substituting the first, second, and third eigenvalues into Equation 2. A Mathcad function was created based on Equation 1 above. The function for a two-term approximation to the exact solution is

$$\theta_2 = A_1 \cdot e^{-\lambda_1^2 \cdot \tau} \cdot \frac{\sin(\lambda_1 \cdot X)}{\lambda_1 \cdot X} + A_2 \cdot e^{-\lambda_2^2 \cdot \tau} \cdot \frac{\sin(\lambda_2 \cdot X)}{\lambda_2 \cdot X}$$

It is generally accepted[1] that a solution based on just one term of this series is accurate within 2% for values of dimensionless time, τ, greater than 0.2. The graph in Figure 2 shows the dimensionless temperature at the center of the sphere for τ less than 0.2. Clearly, the one-, two-, and three-term approximations to the exact solution converge as τ approaches 0.2 from the left.

The dimensionless temperature should approach one as τ approaches zero. All three approximations deviate from one when τ is sufficiently small. As the graph shows, the one-term approximation diverges from the other two at values of τ less than 0.2. The two-term solution appears to be accurate to about τ = 0.06, and the three-term solution appears to be accurate to about τ = 0.04. As more terms are added, the solution can be expected to be accurate to even smaller values of τ. This is of questionable merit, because at τ = 0.06, the dimensionless temperature is already close to 0.98 for both the two- and three-term approximations.

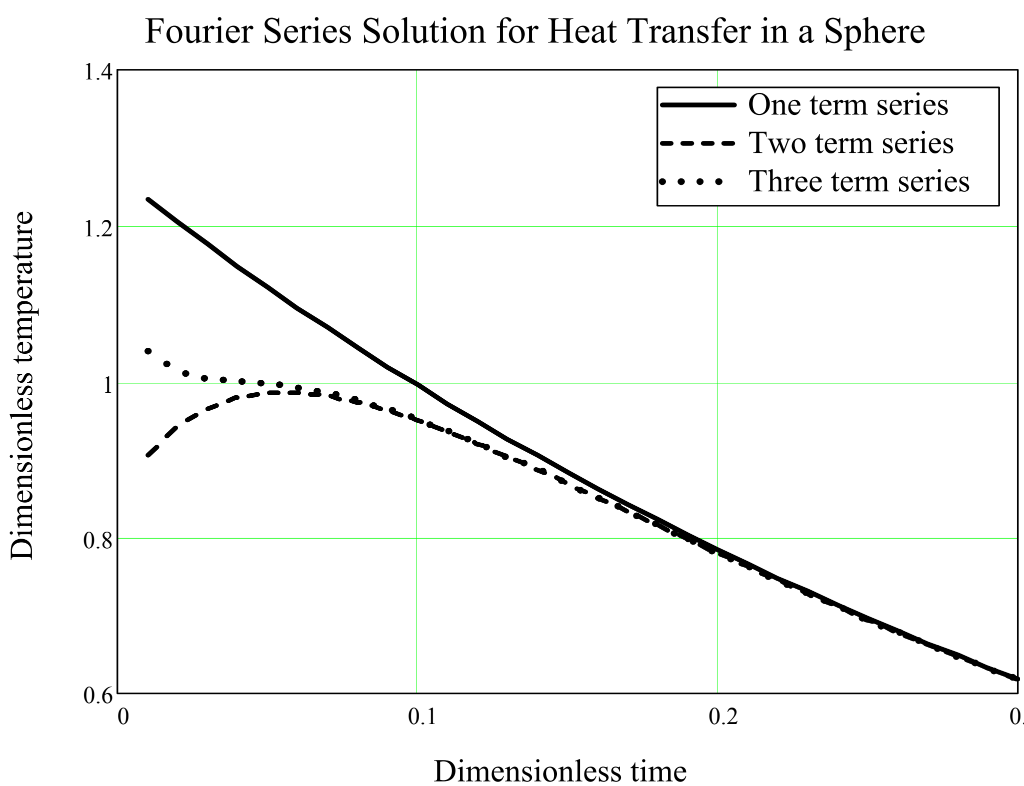

Figure 2

One-, Two-, and Three-Term Approximations to Center Temperature of Sphere

The Lumped Capacity Approximation

The simplest of the approximate methods is called the Lumped Capacity Method. In this method, it is assumed that conduction within the sphere is much more rapid than convection from its surface. As a result the sphere is considered to be at a uniform temperature. A solution is obtained by equating the rate of change of internal energy of the sphere to the rate of convection from the surface. The differential equation is:

$$mC_P \frac{d\left(T - T_{fluid}\right)}{dt} = hA_s\left(T - T_{fluid}\right)$$

where m is the mass, C_P is the heat capacity, h is the convection coefficient, and A_s is the surface area of the sphere. The familiar solution is:

$$\frac{T - T_{fluid}}{T_{initial} - T_{fluid}} = e^{-\frac{hA_s}{m \cdot C_P}t}$$

This solution is considered valid if the Biot number is less than 0.1.

In this case, the Biot number is defined as h·L/k, where the representative length L is defined as the volume divided by the surface area. In the case of a sphere, this is one third of the radius. In the denominator, k is the thermal conductivity of the sphere.

The Heisler Chart Approximation

This approximation is based on the first term of the Fourier series exact solution. Charts of this solution are widely available in heat transfer textbooks and in handbooks. As indicated in Figure 2, the one-term approximation is quite accurate for $\tau > 0.2$.

The Heat Balance Integral HBIM) Approximation

Details of this method are to be found in Chen and Kuo[2]. This method is also considered valid for $\tau > 0.2$. Because the equations for the HBIM approach are quite complex, they are not reproduced here. Interested readers should consult Reference 2 for this information.

A Comparison of Methods

All of the methods above have been applied to a sphere of radius 2.75 cm with the following properties: k = 0.632 W/m/ºC, ρ = 1000 kg/m^3, C_P = 1.0 kJ/kg/ºC. The sphere is initially at a uniform temperature of 8ºC. At time zero it is submerged in a fluid at 100ºC with convection coefficient 22 W/m^2/ºC. The temperatures at the center and surface of the sphere after 3 and 20 minutes are to be found.

For the conditions stated above, the Biot number is too large by about a factor of three, so the Lumped Capacity approximation should not be considered valid at either time. The value of τ is 0.15 at three minutes and 1.0 at 20 minutes, so the Heisler Chart and Heat Balance Integral approximations should be valid at 20 minutes but invalid at 3 minutes.

Table 1 shows a comparison of results after 3 minutes for the following methods: Lumped Capacity, Heat Balance Integral, one-term Fourier series (Heisler Chart), two-term Fourier series, and three-term Fourier series. As noted above, Lumped Capacity, Heisler Chart, and HBIM are not valid. Despite that, Heisler Chart and HBIM agree quite well with the two- and three-term approximations to the exact solution.

Table 1
Sphere Temperatures (ºC) at 3 Minutes.

	Lumped Cap	HBIM	1-Term	2-Term	3-Term
Center	40	18	19	20	20
Surface	40	47	47	47	47

Table 2
Sphere Temperatures (ºC) at 20 Minutes.

	Lumped Cap	HBIM	1-Term	2-Term	3-Term
Center	95	89	89	89	89
Surface	95	93	93	93	93

As expected, Lumped Capacity does not give a useful solution.

Table 2 shows a comparison of results after 20 minutes for the same methods. The value of τ is 1.0, so Heisler Chart and HBIM are valid. All methods except Lumped Capacity agree within one degree Celsius. Even Lumped capacity is fairly close with an error of five or six degrees Celsius at the center and two degrees Celsius at the surface. Of course, the asymptotic solution as time goes to infinity is a uniform temperature of 100ºC throughout the sphere. Thus the 20 minute case is not particularly demanding.

Conclusion

Calculations for the three-term Fourier series solution for transient heat transfer in a sphere subject to a convection boundary condition are quite tedious. Equivalent calculations for the Heat Balance Integral Method are even more so. The author has created Mathcad templates for both approaches and made these templates available to students on a course Web site. Thus there is no need for students to "reinvent the wheel" in order to explore these methods.

The numerical values presented above are taken from an assignment in which students compare methods for solving transient heat transfer. This assignment is used to help them gain an appreciation for the range of applicability of several approximate solution techniques.

Bibliography

1. Cengel, Y.A., Heat and Mass Transfer: a Practical Approach, 3rd edition, McGraw-Hill, 2007, p. 230.

2. Chen, R. Y. and T.L. Kuo, "Closed Form Solutions for Constant Temperature Heating of Solids", Mechanical Engineering News, vol. 16, no. 1, Feb. 1979, p. 20.

Biographical Information

Edwin G. Wiggins holds BS, MS, and Ph.D. degrees in chemical, nuclear, and mechanical engineering respectively from Purdue University. He is the Mandell and Lester Rosenblatt Professor of Marine Engineering at Webb Institute in Glen Cove, NY. He is a past chairman of the New York Metropolitan Section of the Society of Naval Architects and Marine Engineers (SNAME) and a past regional vice president of SNAME. As a representative of SNAME, he served on the Technology Accreditation Commission, the Engineering Accreditation Commission, and the Board of Directors of the Accreditation Board for Engineering and Technology (ABET). A Centennial Medallion and a Distinguished Service Award recognize his service to SNAME.