SEARCH AND RESCUE CARRY THROUGH MATHEMATICAL MODELING PROBLEM

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Abstract

In the Naval Postgraduate School's Department of Defense Analysis, we improve the mathematical preparation of our experienced professionals by enabling them to make informed, rational decisions based on quantitative reasoning. This paper addresses an in-class exercise that combines a solid review of algebra and trigonometry topics in order to do mathematical modeling, problem-solving, and a computer simulation in Excel. This problem demonstrates the importance of basic mathematical concepts from algebra and trigonometry to solve current, relevant problems. The connections between the mathematics they need and its application to solve real problems becomes clear to our students. Technology is essential to facilitate inquiry and experimentation as part of the modeling and problem solving process.

Introduction

In mathematics, you don't understand things. You just get used to them.
  – Johann von Neumann

Our students are mostly field grade military officers from all services both U.S. and International as well as high grade civilians of the federal government from the Department of Defense. In our department, Defense Analysis these highly motivated students take a three course sequence in mathematical modeling for decision making. We use Excel as our technology tool. We use Excel because all these students will have Excel on their computers during the service and thus, can use Excel to help in quantitative decision making.

Conflict is as ancient as humankind, and its study requires an analytic disposition. For that reason, the officers who matriculate through the Department of Defense Analysis undertake a mathematics sequence that enhances their abilities to use quantitative reasoning. The overall intent is to make them better informed decision makers. The students' three course sequence includes deterministic mathematical modeling for decision making, probability and statistics, and game theory. The prerequisite for this three course sequence is college algebra. For most students, it has been some time since they have seen or used algebra skills let alone trigonometry skills. We will quickly discuss our courses before introducing the carry-through problem.

Our first course covers deterministic modeling, including topics such as graphs as models, dynamical systems models featuring a variety of predator-prey and Lanchester scenarios (implemented using Excel spreadsheets), concepts of agent-based modeling, model fitting with least squares, and an introduction to linear programming models in two-variables (graphical) and multi-variable (with the Excel solver). Our second course introduces stochastic modeling. We start with a solid review of statistics and classical probability, including Bayes' Theorem, basic distributions (binomial, Poisson, uniform, exponential, and normal) and their uses in modeling and reliability analysis, and Monte Carlo simulation modeling. Our third course, Models of Conflict, covers a review of expected value for use in decision trees, linear programming as a decision tool, and game theory.

A major portion of our efforts is using technology to help in quantitative decision making. We decide on Excel because it is available to these students when they return to their professional jobs. Students find the tasks in most of our areas of study to be tedious and take their focus from interpretation to mechanics. The use of Excel and templates that we prepare for our students allow the computer to accept the students inputs and processes the tedious calculations allowing the students to put the information together in a coherent manner and make interpretations of Excel's results to help in making decisions.

The span of mathematical abilities found among our students reflects national trends. A significant portion of our students exhibit deficiencies in basic algebra skills. College algebra constitutes the only prerequisite for our courses, but many of our students have not seen the material for as little as seven and as long as twenty years. To our advantage, the students are nearly uniformly motivated to learn, and most quickly re-acquire the necessary algebraic proficiency.

The following carry through problem requires the use of many important algebra skills and pre-calculus skills, so we introduce the problem on the first day of the first course and continue it throughout the first two courses.

Military to the Rescue

We pose the following problem scenario (a modification from a 1988 Mathematical Contest in Modeling problem) to the students on their first day of class.

Two observation posts 5.43 miles apart pick up a brief radio signal. The sensing devices were oriented at $110°$ and $119°$ respectively when a signal was detected. The devices are accurate to within $2°$ (that is $\pm 2°$ of their respective angle of orientation). According to intelligence, the reading of the signal came from a region of active terrorist exchange, and it is inferred that there is a boat waiting for someone

to pick up the terrorists. It is dusk, the weather is calm, and there are no currents. A small helicopter leaves a pad from Post 1 and is able to fly accurately along the $110°$ angle direction. This helicopter has only one detection device, a searchlight. At 200 ft, it can just illuminate a circular region with a radius of 25 ft. The helicopter can fly 225 miles in support of this mission due to its fuel capacity.

We ask the students to do the following in the first course in an effort to review such topics as slope, lines, equations of lines, coordinate systems, intersections of lines, areas, angles, trigonometry, and percentages.

1. Sketch the area of the search region enclosed by the $\pm 2°$ accuracy in deviation of the two sensing devices.
2. Find equations for the four lines bounding the region.
3. Find the coordinates of the points of intersection.
4. Find the area of the region.
5. If the upper left corner of the search region and the lower right corner set limits on a search region, then what fraction of the rectangle is the search region's area?

In the second course covering probability and statistics, we add the following.

6. Generate 1000 random starting points that could represent the location of the terrorist boat (choose an appropriate grid). Include a plot of the generated data.
7. Assume there is sufficient fuel to search 16 square miles. Draw a square 4 miles on a side around some arbitrary location in the search area. How many of your random starting points are inside the square?
8. What percent of the random starting points are inside the box?
   Assume no evasive action on the part of the terrorist boat.

9. Describe the region in which you have 90% confidence for the starting point of the boat. Include a sketch or graph.
10. Estimate the area of the region.
11. Determine a "sweep width" for your search. Explain how you determined the sweep width. Include a sketch.
12. How much of the area can you sweep during the 225 miles that the helicopter can fly?
13. Discuss several search strategies for the helicopter.
14. Pick *one* of your search strategies.
 a. Estimate the probability that your strategy finds the boat.

 b. How many of the typical starting points were you able to find?
15. How many search helicopters would you need to have a 95% chance of finding the target?

Course 1: Modeling Process and Deterministic Models

We ask our students to implement a modeling process using the eight steps described by Fox (et al, 2009). In short, the students take a real world system, apply simplifying assumptions, then solve it so that they might study its effects. This modeling process is illustrated in Table 1 and Figures 1 and 2.
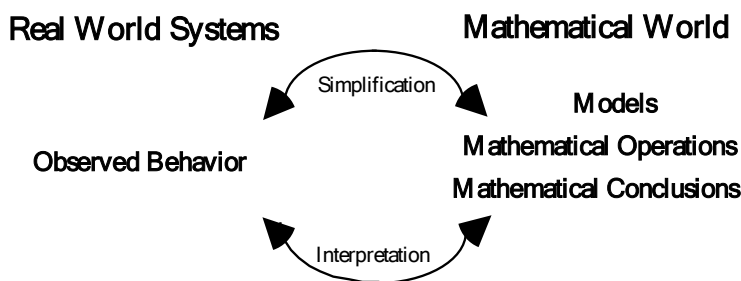

Figure 1: Mathematical Models and the Real World.

Step 1. Understand the problem or the question asked.
Step 2. Make simplifying assumptions.
Step 3. Define all variables.
Step 4. Construct a model.
Step 5. Solve and interpret the model.
Step 6. Verify the model.
Step 7. Identify the strengths and weaknesses of our model.
Step 8. Implement and maintain the model for future use.

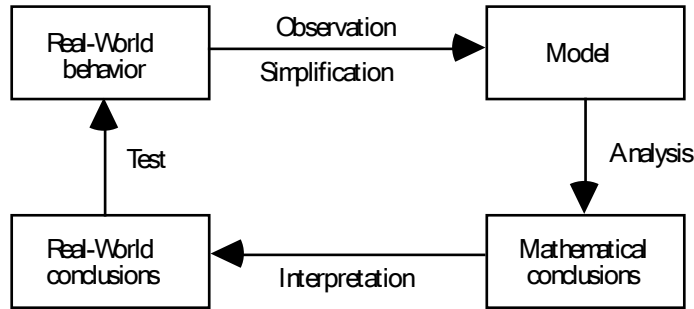Table 1: Steps in the Mathematical Modeling Process.

Figure 2: Schematic of the mathematical modeling process.

The following review topics are covered during this portion of the problem.

- Graphing in 2-dimensions
- Angles
- Coordinates
- Slope
- Lines and their equations
- Intersections of lines
- Distance formula
- Areas formulas
- Trigonometry review: Law of Sines and Cosines

The review is in context of solving a bigger problem, so there is motivation to review these subjects.

The students identify the problem statement: they are to build a mathematical model that relates a search strategy to the chances of finding a target. To better understand the process they consider modeling assumptions as well as defining needed variables (and classify them as either independent or dependent variables). We suggest using graphs whenever possible. So they first construct a sketch of the rescue area, using the x-axis as an east-west axis, and the y-axis as the north-south axis.

Issues related to Figure 3:

1. The observation posts sightings are not perfectly accurate; their accuracy is ±2°.

2. Students need to find the area where target might be located.
3. We need to find coordinates of triangle.

To help solve the problem an assumption has to be made: where to locate the origin? For our purpose, we will assume that the right-most (furthest east) observation post has coordinates (0, 0). The task then becomes: how to find the coordinates of the triangle.

Using the initial sighting angles, Angle 1 = 61°, Angle 2 = 110°, Angle 3 = 9°, as well as the fact that the distance between two sites is 5.43 miles (the side opposite to the 9° angle), and labeling Point 1 (0,0), Point 2 (0,5.43), we update the original sketch as we seek to find the coordinates of Point 3.

To solve this, we review a portion of trigonometry, specifically the Law of Sines. Applied to Figure 4, the students obtain

$$\frac{\sin(Angle3)}{5.43} = \frac{\text{Sin}(Angle\ 2)}{x} = \frac{\text{Sin}(Angle\ 3)}{y}$$

Since *sin 9° / 5.43 = .02881*, the lengths of the other two sides are therefore

Sin 110° /x = .02881 => x = 32.617 miles and *Sin 61° /y => y = 30.358* miles.

In the current situation, the students need to find the lengths l_1 and l_2, shown in Figure 5.

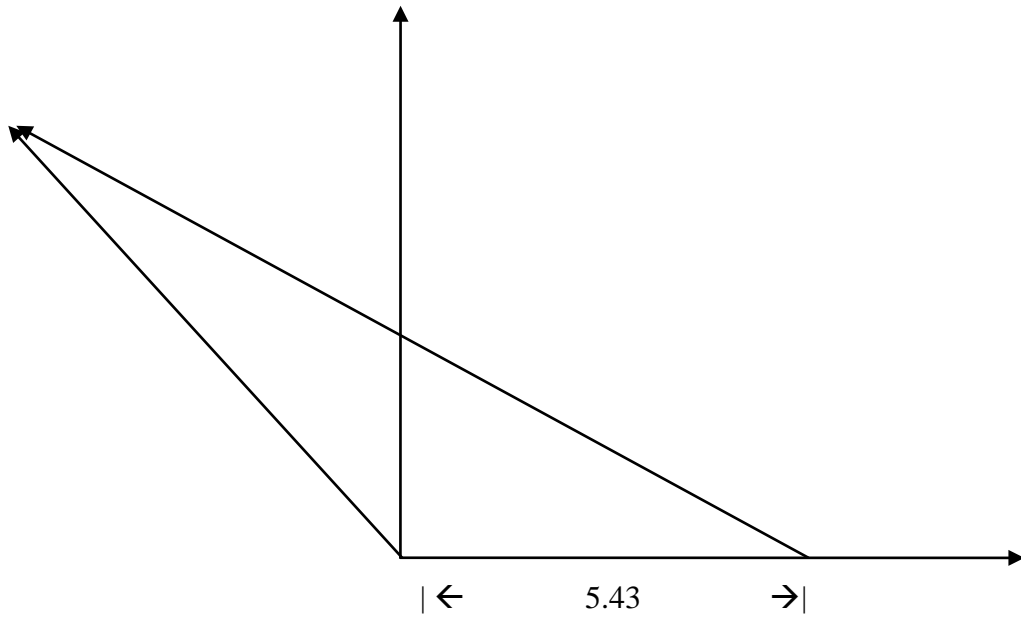Figure 3:   Plot of observation posts and directions to observations.

Point 3

Angle 3

y

x

Angle 2   Angle 1
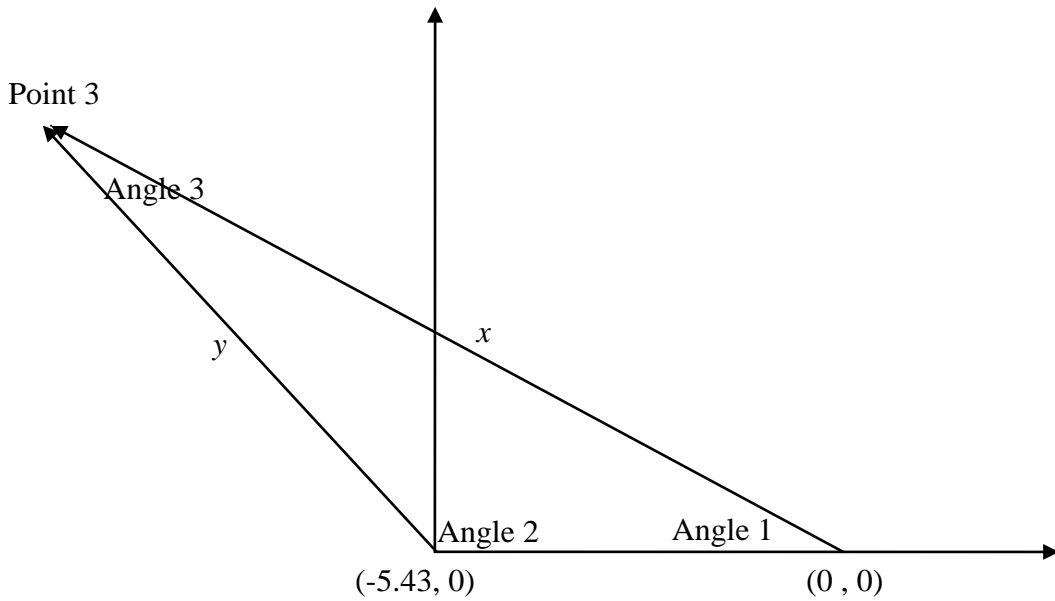
(-5.43, 0)   (0 , 0)

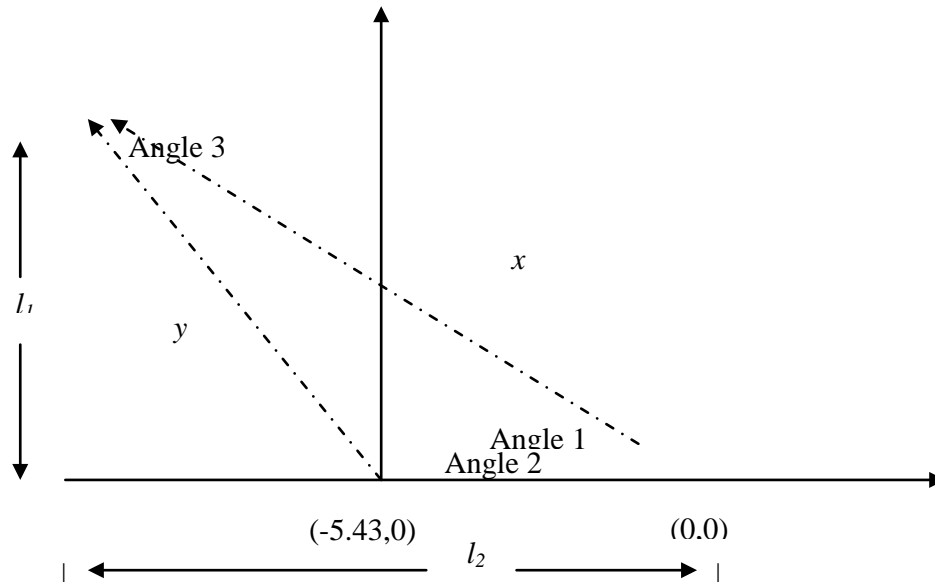Figure 4: Updated sketch of the observations showing the angles.

Figure 5: Updated sketch depicting unknown lengths needed to represent Point 3 relative to the origin.

Since we have a right triangle, we again employ basic trigonometry to find the length of the side opposite Angle 1. That is, knowing the length of side x, we can find l_1 from the relationship $l_1/x = \sin(\text{Angle 1})$. Thus, $l_1 = x * \sin(\text{Angle 1}) = 32.617 * \sin(61^\circ) = 28.52$ miles. Now, from the Pythagorean Theorem we know that $32.617^2 = l_1^2 + l_2^2$. Letting $l_2 = (l_3 + 5.43)$, then $30.358^2 = (28.52)^2 + (x+5.43)^2$, from which we obtain $l_3 = -10.413$. So the coordinates of Point 3 relative to the origin are *(-10.413,28.52)*.

Next, we must find the equations for two lines that intersect:

Use point-point method or the point-slope form, which requires that the students first find the slopes. Equations of lines, their slopes, and the intersection of lines are critical skills to success in this first course, so this portion of the problem is quite important.

$(y-y_1) = $ m $(x-x_1)$ where $m = $ slope $= (y_2-y_1)/(x_2-x_1)$

Line 1: Use as point 1, $(x_1,y_1) = (0,0)$ and slope1 $= 28.52/(-10.413) = -2.73888$. Thus, $y = -2.73888x$

Line 2. Use as point 2, $(x_2,y_2) = (-5.43,0)$ & slope 2 as (28.52/(-10.413-5.43)
(y-0) = slope 2(x+5.43) slope 2 (28.52/(-10.413-5.43)
Thus,
y = -1.804(x-5.43) = -1.804 x + 9.796

Next, we repeat this process when we are off by 2 degrees on 110° and 119° use 108° and 112° and 117° and 121°. After some work, students should obtain the following equations of lines:

$a := -2.4709 \cdot x;$

$b := -2.73888\ x$

$c := -3.0776 \cdot x;$

$d := -1.6642 \cdot x + 9.037;$

$e := -1.804 \cdot x + 9.796;$

$f := -1.9626 \cdot x + 10.656;$

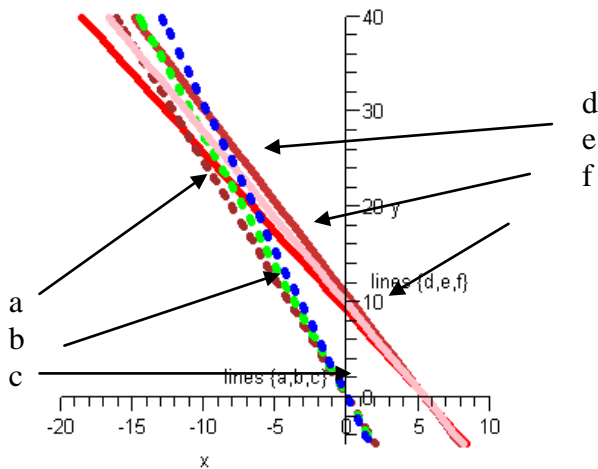We graph those lines to see what is happening in terms of their intersections and the region that is bounded:

Figure 6: Graph of all the lines of possible sight.

Figure 7: The diamond shaped region of interest.

The common intersection defines the feasible region or area of concern. Note in Figure 7 that the region is shaped like an elongated diamond (yellow shaded region). This region is formed by the intersections of lines *a, c, d* and *f.*

The shaded region is the appropriate search area. In future classes on search techniques, we will need the area of the region. The area of the diamond (general quadrilateral) can be found using the formula below:

$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{4}(ac+bd+pq)(ac+bd-pq)}$$

where *a, b, c* and *d* are the four sides of the quadrilateral with *a* & *c* and *b* & *d* as the opposite sides, *p* & *q* are diagonals, and *s = (a+b+c+d)/2.*

There is more work to do. The students need to find the coordinates of the corners of the diamond, which come from solving two equations in two unknowns – repeatedly (this is another skill that carries through the course sequence). Then they can use those coordinates in the distance formula to find the length of *a, b, c, d, p*, and *q*. We ask the students to solve the
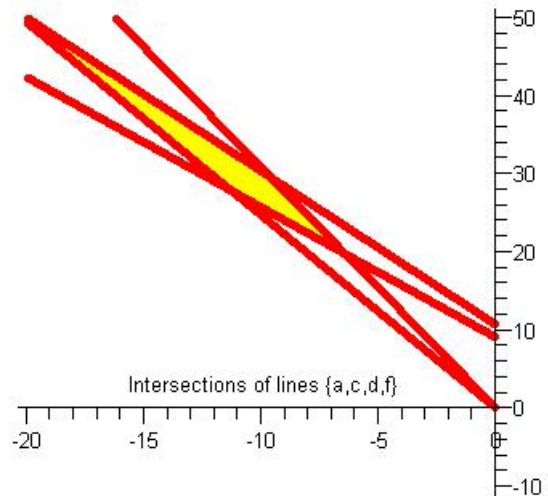
two equations by hand as a means of reinforcement; however, if one is adept and has a computer algebra system available (Maple, for instance), then the coordinates can be found somewhat more quickly. Leaving out the details, we find the coordinates of the diamond's corners:

*(-6.3938,19.6776), (-20.964, 51.79),
(9.5570,29.412) , (-11.292,27.68)*

We graph the points and connect them to obtain the diamond shape in Figure 8.

Again, we use the distance formula to find the lengths of the sides of the quadrilateral.

The distance between (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This distance is based upon the Pythagorean formula, $a^2 + b^2 = c^2$.

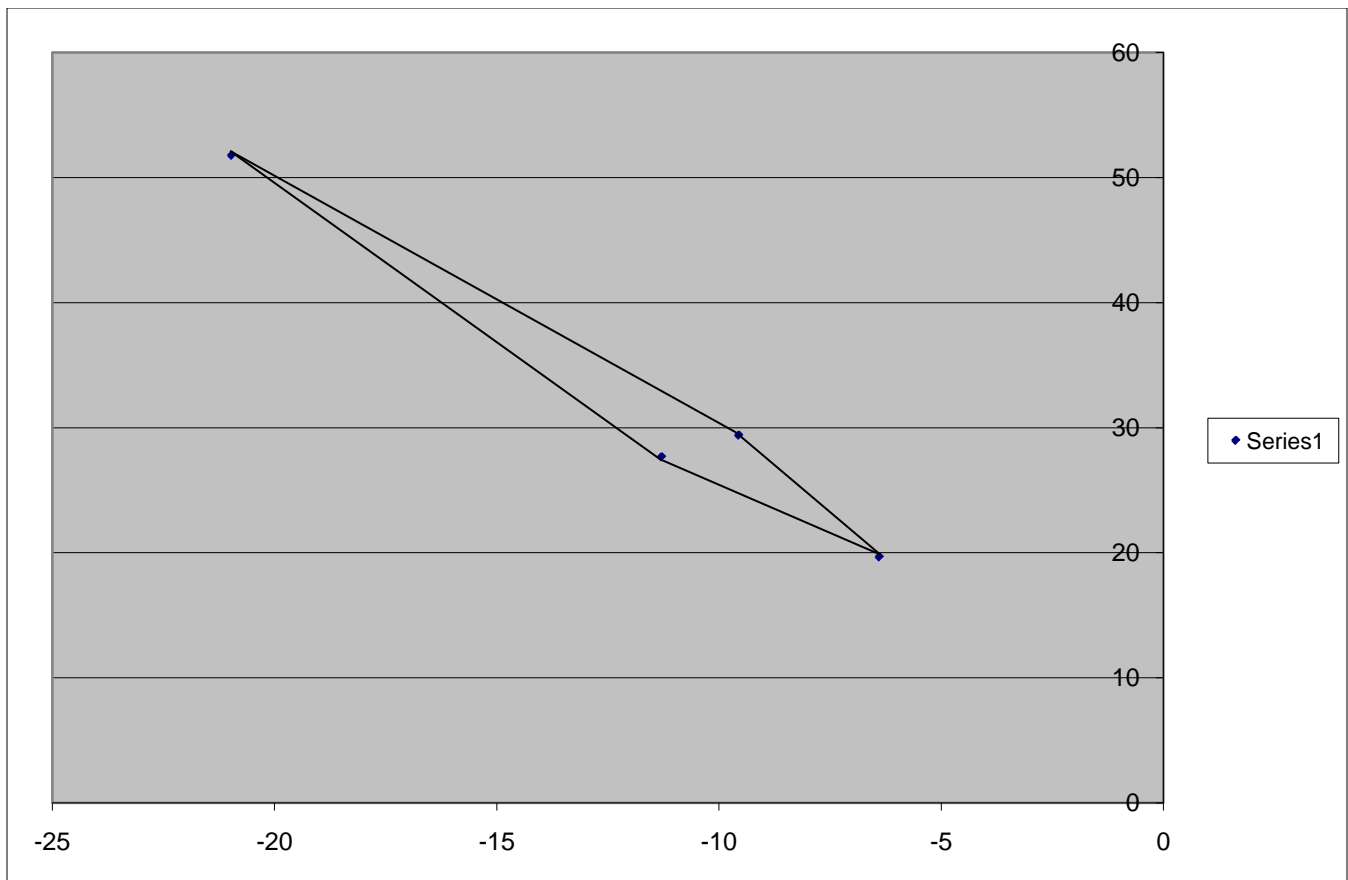Using the distance formula, we find the lengths of the sides and the diagonals (p and q) are

Figure 8. The diamond shaped region to scale.

a= 9.38, b=10.25, c=25.12, d= 25.97, p=2.452, q = 35.259

We find s for use in the area formula using, *s = (a+b+c+d)/2* . We find, by substitution and simplification, that the value of *s* is *35.3566*.

Area =

$$= \sqrt{\begin{aligned}&(s-a)(s-b)(s-c)(s-d)-\frac{1}{4}(ac+bd+pq)(ac+bd-pq)\end{aligned}}$$

$$= \sqrt{\begin{aligned}&(35.3566-9.38)(35.3566-10.25)(35.3566-25.12)\\&(35.3566-25.97)-\frac{1}{4}(10.24\cdot25.97+9.39\cdot25.12+\\&2.452\cdot35.259)(10.24\cdot25.97+9.39\cdot25.12-\\&2.452\cdot35.259)\end{aligned}}$$

= 40.86.

Thus, the search area is 40.86 mile^2. The rectangular region bounded by the upper-left and lower-right corners is 20.964 * 51.79 = 1085.73 mile^2, which makes the search region approximately 3.7% of that larger, rectangular area.

Course 2

Course 2 covers modeling with probability and statistics. We teach such topics as basic statistics, classical probability, random variables and distributions, confidence intervals, hypothesis tests, survey analysis, Monte Carlo simulations, and special applications to targeting and searches. It is during the last block that we return to the search problem. We include arbitrary and random searches in our brief

survey of Search Theory techniques, and we apply them to our problem.

Assume that we use the helicopter mentioned. We choose a speed of 60 MPH (it was not specified), and since the helicopter can stay up for 225 miles then the time available to search, T, is T=225/60 = 3.75 hours. Assume also that the search light illuminates a circular region of radius 25 feet and that if the target gets inside the searchlight's radius, then the target is found with probability 1.

Now, 25 feet is 0.0047 miles (0.0047=25/5280). This gives us a lateral detection scheme as follows:
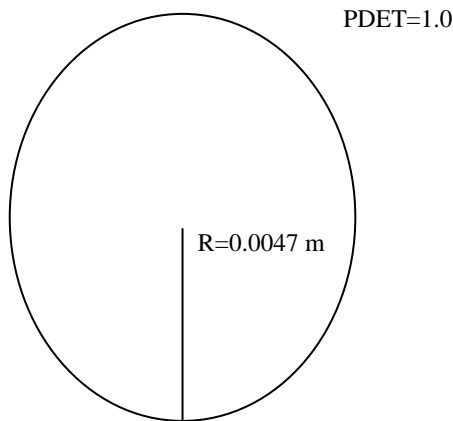
This is a circle whose diameter, 2*RMAX, and whose height (probability) is 1.0.

$$W = 2*Rmax = 2*.0047 = .0094$$
$$A = 40.86 \ miles^2$$
$$V = 60 \ mph$$
$$T=3.75 \ hours$$

We can find *S*, defined to be

$$S= VW/A = (60*.0094)/40.86 = 0.0138$$

In an arbitrary search, methodology, the probability of detection by time t ,PDET(t), is
PDET(3.75) = S*t = .0138*3.75 = .0517

If we use a random search, then PDET(3.75) = 1-e$^{(-3.75*.0094)}$= 1-e$^{-.0517}$=.0503

Both probabilities are quite low.

Can we do better with other search patterns? We will examine Monte Carlo simulations as an option.

Let us assume that the helicopter can search 16 square miles. A simple search area might be depicted by a 4 mile by 4 mile box in the search region. We choose to center the box at coordinates (28.5,15.8); however another option is to find the some other location and center a search box there. We then find the intersection of the lines that form p and q (the diagonals of the quadrilateral, defined above), and obtain their coordinates. We then go ± 2 miles to each side.

Line for p: y = x + 38.97
           y = -2.203 x + 5.606

from which we find x = -10.42, y = 28.55, so the center point is (-10.42, 28.55).

The search box square goes from -8.42 to -12.42 in *x* and 26.55 to 30.55 in *y*, as shown in Figure 9.

We might want to try a "new" search method, perhaps one that might be evaluated using Monte Carlo simulation. A Monte Carlo simulation is described in Figure 10 that can provide the necessary probabilities for comparison.

We show a few steps by hand of this simulation.

Count=0
Do *i=1,2*

Generate a random location inside our diamond such as (-11,30).
Determine if our location is inside our search zone. Yes, *Count=0+1=1*.
i=2
Generate another location (-7.5, 22)
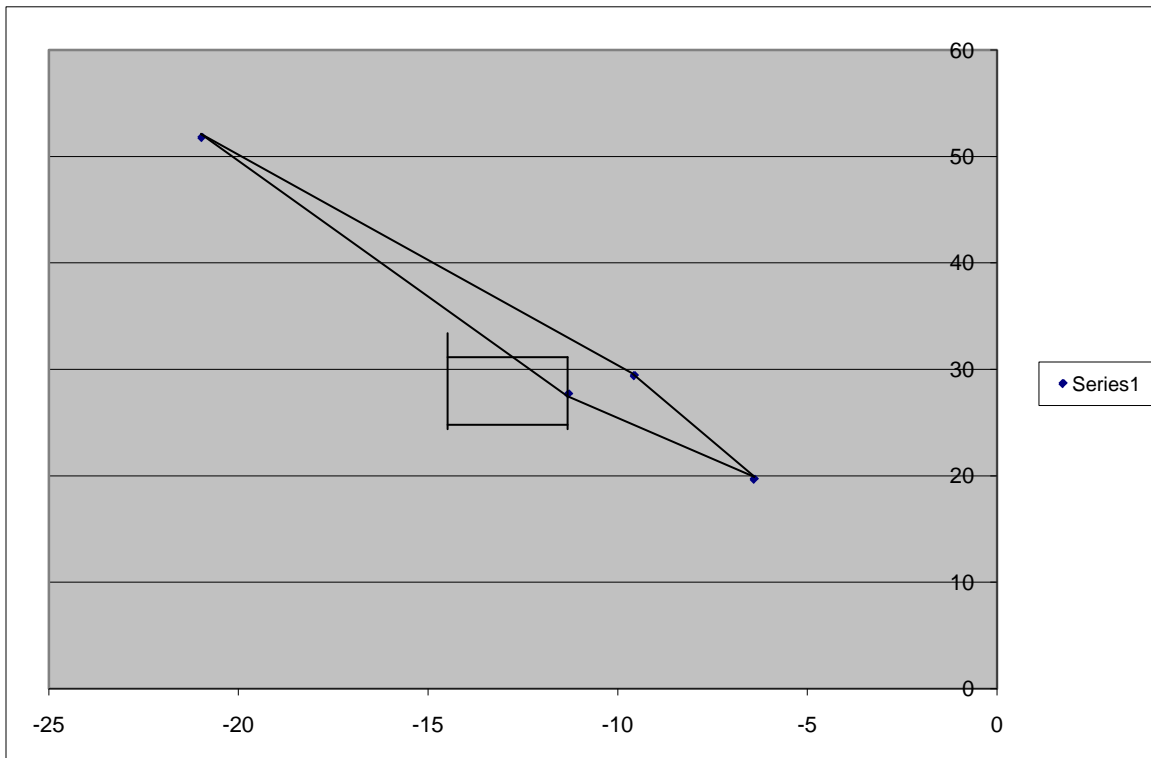No, *Count=1*
P(S) =1/2=50%

Figure 9: Overlay of search grid in our search region.

Algorithm:

INPUT: size of search area, sensor ranges, search range, search style or method, number of trials, regions for random numbers
OUTPUT: Probability of success

Initialize the value of count to 0
Do *i= 1,…,n*
Step 1. Generate a random location for the target that is inside the region .
Step 2. Determine if the target is inside our search area. If yes, *count = count + 1* otherwise *count=count*
Step 3. Calculate: $P(S) = count / N$

Figure 10: Monte Carlo Simulation Algorithm.

Now imagine trying to do this for a large number of trials. Probability is based upon the law of large numbers. Literally thousands upon thousands of trials should be run. The computer makes this more accessible.

We use EXCEL to perform the simulation. We choose a random target location inside the diamond, and then we apply a search method to see if it gets close enough to the random point to find it. In this case, we either found the target (*Value = 1)* or we did not find the target (*Value =0)*. Now, we have Excel repeat this process a 50,000 times in very little time (under a few minutes). We then count the number of times the target is found (*Value = 1*) and divide that number by *n*, the number of times we ran the simulation. This is our probability of success, *P(success)*.

Statistics from Simulation	
Mean	0.18058
Standard Error	0.0000068692
Median	0.183
Mode	0.188
Standard Deviation	0.015361752
Sample Variance	0.000235983
Kurtosis	0.12035735
Skewness	-0.002928607
Range	0.082
Minimum	0.14
Maximum	0.222
Sum	18.058
Count	50000
Confidence Level(95.0%)	0.003048105

In our example, we ran the simulation 50000 times and found the target approximately eighteen percent of the trails. A 95% confidence interval for the mean is:

$$0.18058 \pm 1.644 \cdot \frac{0.015361752}{\sqrt{50,000}}$$

$$0.18058 \pm 0.0001129$$

Devising search experiments led to many interesting discussions of possible search patterns with the intent of improving chances of finding the target. For example, it is possible for students to print out a plot of the possible locations of the target. Then using acetate, students can create their own search pattern of width, *W*. For instance, they can lay acetate over the random points (all to scale). They can count the number of points that touch the search pattern and divide by N to obtain their probability of success.

Conclusion

Not only is this problem a very interesting modeling problem, but the wide spectrum of review topics adds a richness that students need. We find that both students and faculty appreciate this type realistic problem that combines quantitative reasoning to a realistic scenario. It generates class room discussions and shows the importance of mathematics in solving real problems.

References

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Biographical Information

Dr. William P. Fox received his BS degree from the United States Military Academy at West Point, New York, his M.S. at the Naval Postgraduate School, and his Ph.D. at Clemson University. He has taught at USMA, at Francis Marion University, and is currently a professor at the Naval Postgraduate School. He has authored two textbooks on mathematical modeling. He has over one hundred technical and educational articles, presentations, and workshop presentations. He serves as the associate contest director for COMAP's Collegiate Mathematical Contest in Modeling (MCM) and serves as contest director for the High School Mathematical Contest in Modeling (HiMCM). His interests include applied mathematics, optimization (linear and nonlinear), mathematical modeling, statistical models for medical research, and computer simulations.

Dr. Michael Jaye received his BS degree from the United States Military Academy at West Point, NY, his M.S. and Ph.D. from Rensselaer Polytechnic Institute, Troy, NY. He is an associate professor at the Naval Postgraduate School. He serves as a judge for both the Collegiate Mathematical Contest in Modeling (MCM) and for the High School Mathematical Contest in Modeling (HiMCM). His interests include applied mathematics, modeling, differential equations, agent based modeling, and computer simulations.