USING EXCEL TO IMPROVE UNDERSTANDING OF CALCULUS BASED TECHNIQUES IN FLUID MECHANICS AND DEVELOPING EXCEL BASED NUMERICAL TECHNIQUES WHEN CALCULUS PROVIDES NO CLOSED FORM SOLUTION

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Introduction

The concept described in this paper was used in two courses, namely a fluid mechanics course and in an advanced engineering mathematics course.

In the fluid mechanics course, the technique was used to illustrate the concept of Computational Fluid Dynamics (CFD). The technique illustrated the application of calculus in modeling fluid systems and then presented a technique for numerically solving the resulting equation when no closed form calculus based solution exists.

In the advanced engineering math course, the example was used as a physical illustration of application of calculus in modeling actual physical systems. In the advanced engineering mathematics course the equation was setup using physical and mathematical reasoning, and then solved by using Differential Equation Solution techniques, and it was shown that only numerical techniques can be used when nonlinear elements are present.

In both classes, a component of homework and exams was using EXCEL to solve problems of this nature.

Concept

Fluid flow analysis in a piping system is governed by the principle of conservation of energy. In the absence of pressure difference between the exposed surfaces of fluid flow, every fluid particle has energy from two sources, namely energy due to fluid height (potential energy), and kinetic energy (velocity of fluid flow). Friction takes energy out of the system.

It is shown that when friction effects are excluded, the conservation of energy approach results in a differential equation for calculating rate of fluid flow in a piping system that has an exact solution. However, every differential equation can be solved by numerical techniques. The paper demonstrates that the results obtained by classical solution techniques and those obtained by numerical techniques are the same, and thereby demonstrating the accuracy of numerical techniques.

The conservation of energy technique is then used to develop a differential equation for the system when friction effects are included in the formation of the relationship. Because of nonlinear nature of friction forces in fluid mechanics, this differential equation does not have a closed form calculus based solution and it can only be solved by numerical techniques. Solution of this equation using numerical techniques is then demonstrated.

EXCEL is used to implement the numerical techniques for all the examples presented in this paper.

Nomenclature

V: Velocity (ft / sec).
g: Specific gravity constant (32.2 ft / sec^2).
h: Elevation (ft).
h_L: Head lost due to friction (ft).

- K: Coefficient for determination of head loss due to friction (dimensionless).
- L: Length of straight pipe (ft).
- D: Pipe inside diameter (ft).
- f: Friction factor for straight pipes (dimensionless).
- N_R: Reynolds number (dimensionless).
- ν: Fluid kinematic viscosity (ft^2/ sec).
- Ɛ: Pipe material roughness (ft).
- Q: Rate of fluid flow (ft^3/sec).
- A_t: Cross sectional area of tank (ft^2).
- A_j: Cross sectional area of orifice where fluid flows out (ft^2)

Technical Discussion

Figure 1 illustrates flow from a tank with falling head.

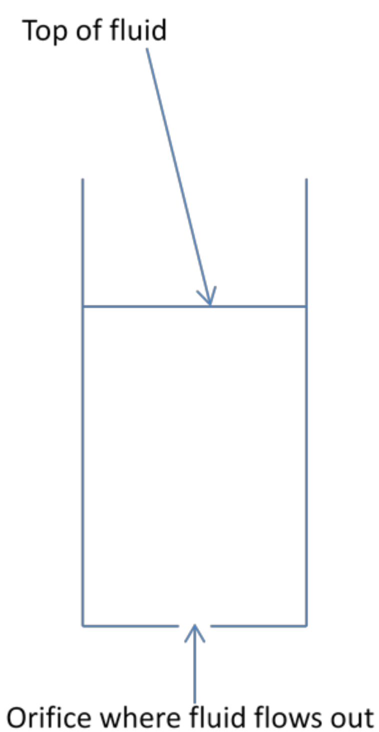

Figure 1: Fluid flowing out freely under gravity action.

For a given depth of fluid h, Torricelli's theorem can be used to calculate the velocity of flow out of the orifice shown in Figure 1. Equation (1)[1] shows this relationship.

$$V_j = (2gh)^{0.5} \qquad (1)$$

The volume flow rate through the nozzle is governed by Equation (2).[1]

$$Q = A_j V_j \qquad (2)$$

In a small amount of time dt, the volume of fluid flowing through the nozzle is governed by Equation (3).[1]

$$\text{Volume flowing} = Q\,(dt) = A_j V_j\,(dt) \qquad (3)$$

Since fluid is leaving the tank, the fluid level is decreasing. During the small time increment dt, the fluid level drops a small distance dh.

Volume of the fluid removed from the tank

$$= -A_t\, dh \qquad (4)[1]$$

Volume flowing (Equation 3) and volume removed (Equation 4) are equal to each other. This results in the equality shown in Equation (5).1

$$A_j V_j\,(dt) = -A_t\, dh \qquad (5)$$

Solving for dt in equation (5) results in Equation (6).[1]

$$dt = \{-(A_t / A_j) / V_j\}\, dh \qquad (6)$$

Substituting for V_j from Equation 1 into Equation 6 results in Equation (7).[1]

$$dt = \{(-A_t / A_j)/ (2gh)^{.5})\}\, dh \qquad (7)$$

Equation (7) is a differential equation that has a known solution. The known solution is shown in Equation (8).[4]

$$t_2 - t_1 = \{2\,(A_t / A_j) / (2g)^{.5}\}\,(h_1^{.5} - h_2^{.5}) \qquad (8)$$

EXCEL can directly be used to make calculations using algebraic Equation (8) for a desired range.

EXCEL can also be utilized to numerically solve equation (7) where small increments dh are used to find small increments for dt. In this technique the output of each calculation step is the input for the next calculation step.[3]

Figure 2 shows the rate of change for fluid height calculated by using the algebraic relationship shown in Equation (8).

Figure 3 shows the rate of change of fluid height calculated by numerically solving the differential equation shown in Equation (7) by using .1 inch height increments and using the output from each step as the starting point for the next step. EXCEL is used to perform this calculation.

Review of Figures 2 & 3 shows that Equations (7) & (8) produce almost identical result if sufficiently small increments are used in Equation (7). For this example, a .1 inch increment is sufficiently small. This illustrates that the numerical technique of solving a differential equation produces the same result that using the algebraic solution obtained from directly solving the differential equation would produce.

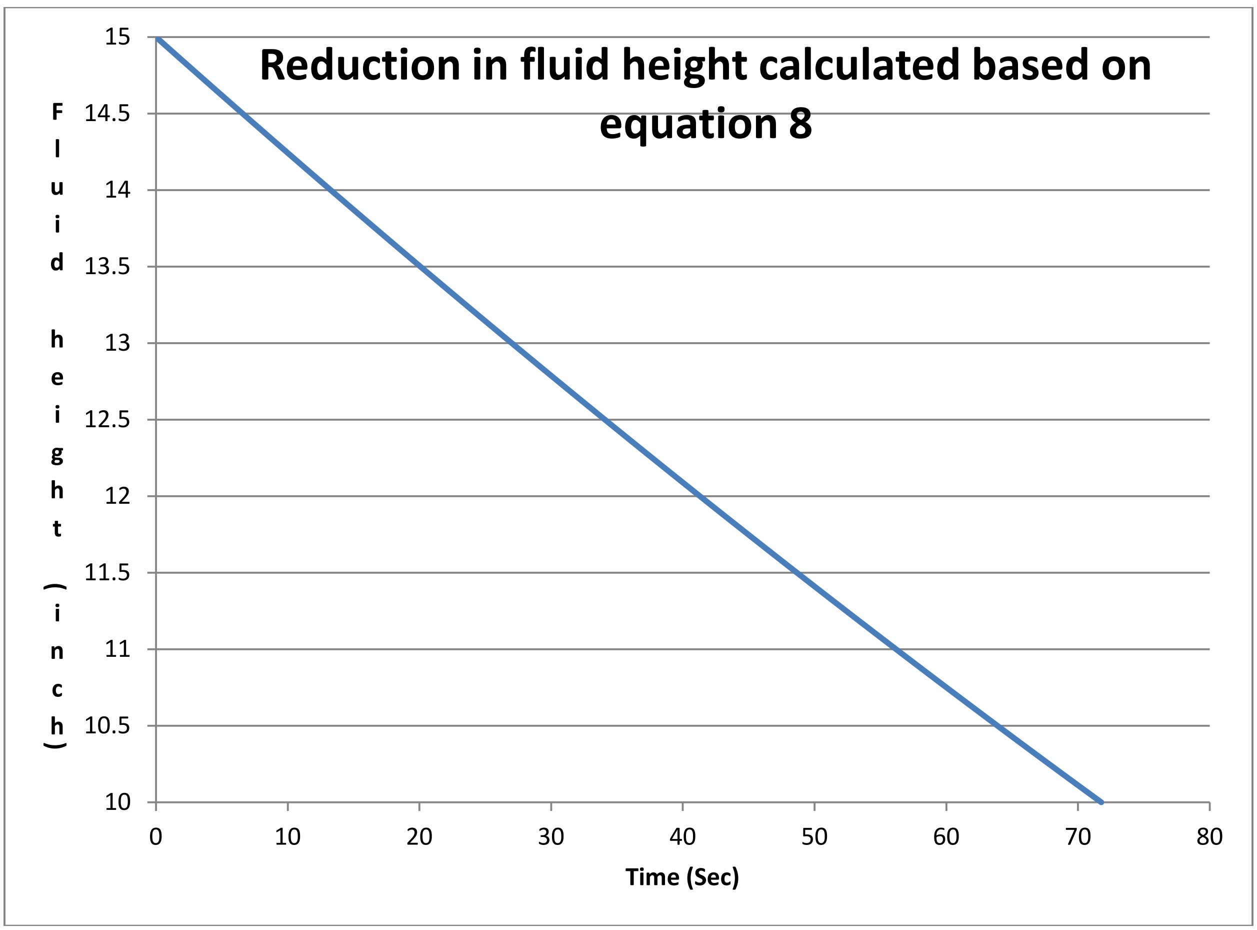

Figure 2.

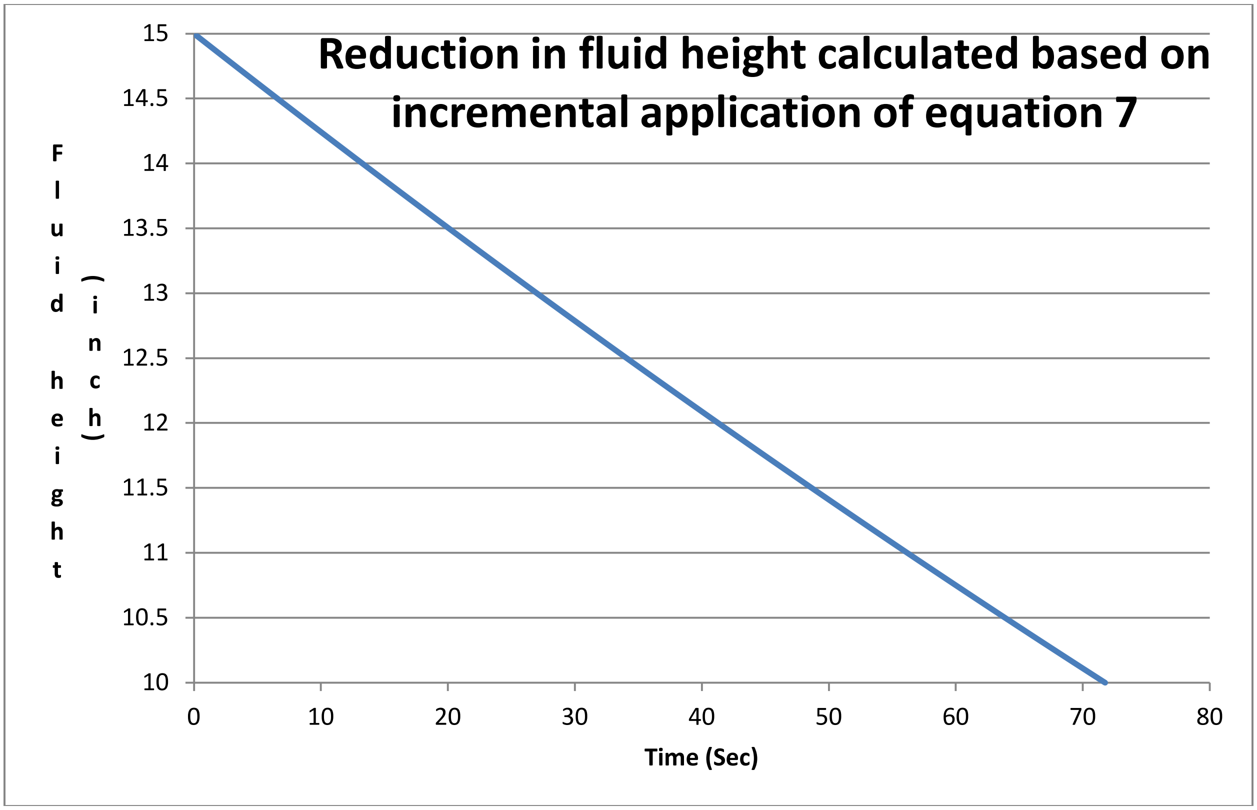

Figure 3.

Equations (7) and (8) were developed assuming the system of Figure 1 had no friction. This assumption resulted in the relationship shown in Equation (7) that had a readily available solution shown in Equation (8). However, assumption of no friction is not a realistic assumption.

If frictional losses are included in the calculations, Equation (7) must be modified as shown in Equation (9).[1]

$$dt = \{(-A_t / A_j)/ (2g(h - h_L))^{.5})\}\ dh \qquad (9)$$

Equation (9) has no exact analytical solution. Consequently, unlike Equation (7), the differential equation shown in Equation (9) has no exact classical solution and it cannot be reduced to an algebraic relationship and can only be solved by numerical techniques. The same numerical technique used for solving Equation (7) can be used to solve Equation (9).

Friction losses in a system can be determined by a general relationship shown in Equation (10).[1]

$$h_L = K\ (V^2 / 2g) \qquad (10)$$

Coefficient K in Equation (10) for a straight pipe can be determined by Equation (11).[1]

$$K = fL / D \qquad (11)$$

Flow in a pipe is either classified as laminar or turbulent. The classification is determined by calculating a quantity referred to as Reynolds number for the flow of fluid in a pipe. If the Reynolds number is less than 2000 the flow is laminar. Between Reynolds numbers of 2000 and 4000, it is not possible to determine the flow classification. For Reynolds numbers greater than 4000, the flow is turbulent. Reynolds number for flow in a pipe can be determined from Equation (12).[1]

$$N_R = VD/ \nu \qquad (12)$$

By having the relative roughness of a pipe (D/Ɛ) and the Reynolds number of fluid flow in a pipe, the friction factor can be read from Moody's diagram. However, use of Moody's diagram is not suitable for computerized analysis of piping systems. In place of Moody's diagram Equations (13) and (14) can be used. Equation (13)[1] is for laminar flow conditions and Equation (14)[2] is for turbulent flow conditions. Generally, fluid flow is considered to be laminar below a Reynolds number of 2000 and turbulent above a Reynolds number of 4000, and flow condition is unknown between Reynolds numbers of 2000 and 4000. Since the theme of this article is numerical techniques, an unknown region will lead to errors. Consequently, an approximate compromise has been chosen by considering flow to be laminar below a Reynolds number of 3000, and turbulent above it.

$$f = 64 / N_R \tag{13}$$

$$f = 0.25 / \{[\log (1 / 3.7 (D/\varepsilon)) + (5.74/ N_R^{0.9})]^2\} \tag{14}$$

Figure 4 is an example of a fluid flow system where the assumption of no friction would not be a reasonable one and consequently Equation (9) instead of Equation (7) must be used.

Equations (9) through (14) are used to recalculate the results shown on Figure 3 using the same incremental numerical technique used for calculating the data of Figure 3. The fluid is assumed to be water at 70 degrees F ($\nu = 1.05 \times 10^{-5}$ ft^2/sec), pipe roughness (ε) is assumed to be 0.0004 ft (pipe is assumed to be coated ductile pipe) and dimensions are as shown on Figure 4. One-third of the results of these calculations along with the results used to plot Figure 3 are summarized in Table 1. (Only one-third is shown in order not to include too long of a table, but at the same time providing enough sample data points).

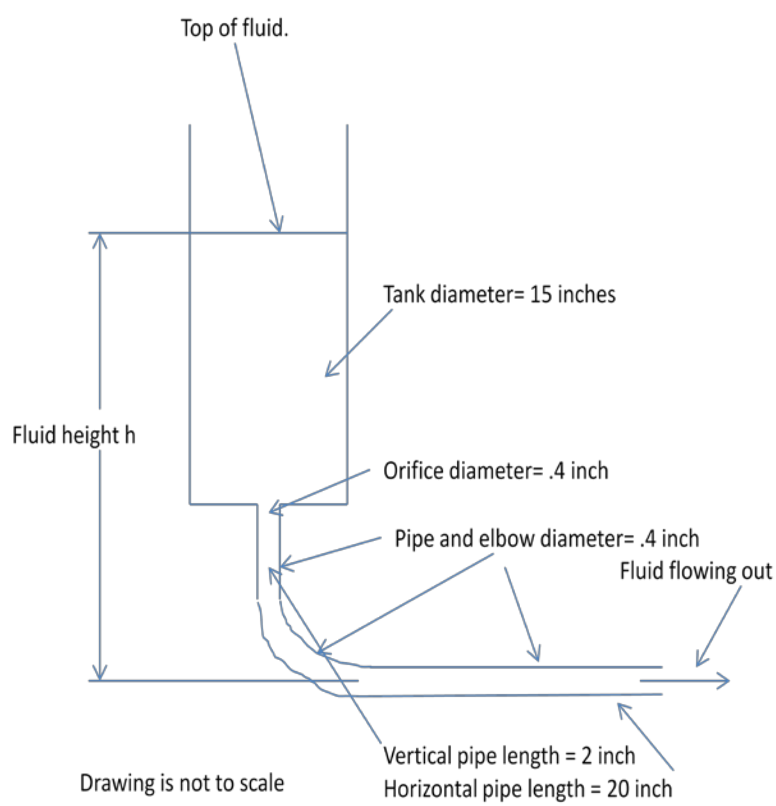

Figure 4: Fluid flowing through a piping system due to gravity

Table 1: Fluid discharge rate for piping system of Figure 4 with and without including friction influence in the calculations.

Fluid height (inch)	Discharge time ignoring friction (sec)	Discharge time accounting for friction (sec)
15	0	0
14.7	3.9	47.2
14.4	7.9	94.9
14.1	11.9	143.1
13.8	15.9	191.8
13.5	20.1	241.1
13.2	24.2	290.9
12.8	29.8	358.1
12.5	34.1	410.0
12.2	38.9	461.1
11.9	42.7	513.5
11.6	47.2	566.5
11.3	51.6	620.3
11.0	56.2	674.8
10.7	60.8	730.0
10.4	65.4	786.0
10.1	70.2	842.9

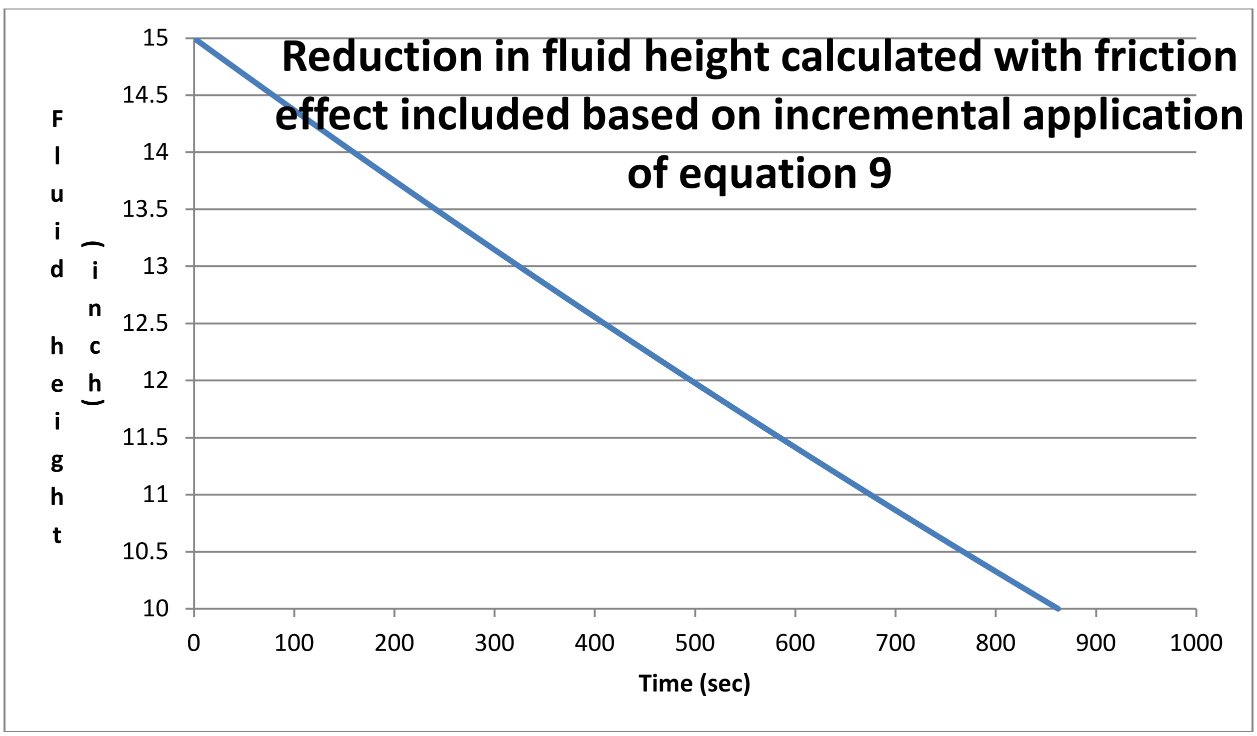

Figure 5.

Figure 5 is a plot that shows the rate of height change when friction is included.

Reviewing the data in Table 1, and comparing Figures 3 and 5 shows the drastic influence of fluid viscosity (friction) on the discharge rate of fluid.

These results clearly show the necessity of using numerical techniques for situations where classical differential equation techniques cannot be used to obtain a closed form solution.

Summary & Conclusion

In this article a fluid system where the fluid flows due to fluid head (potential energy of fluid) was used to demonstrate several concepts. The concepts are as follows:

1. By choosing small increments of time and elevation difference a calculus based approach can be used to setup a differential equation that describes system behavior.

2. When friction effects are excluded from the differential equation, a classical solution for the flow equation exists.

3. The equation with exact solution was solved using algebra and numerical techniques. Comparison of the results demonstrated the accuracy of numerical techniques. EXCEL was used to for both solution techniques.

4. The basic flow equation was then expanded by including frictional losses.

5. A discussion was presented showing that the expanded equation does not have an exact calculus based solution.

6. The expanded equation was then solved using numerical techniques. EXCEL was used to implement the numerical techniques.

In summary, this article demonstrates how EXCEL can be used to solve fluid mechanics systems where classical mathematics cannot provide a solution.

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Biographical Information

Dr. Hagigat is an associate professor in the college of engineering of University of Toledo, and he is teaching engineering technology courses. Dr. Hagigat has an extensive industrial background, and he is continuously emphasizing the practical applications of engineering material covered in a typical engineering technology course.