THE PARTIAL CONFLICT GAME ANALYSIS WITHOUT COMMUNICATION IN EXCEL

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Abstract

We apply linear programming and nonlinear programming to find the solutions for Nash equilibriums in two person nonzero-sum games. Linear programming can easily be used in a 2-Person, 2-strategy game where no pure strategy exists. For games with more than two strategies for a player, we recommend a nonlinear approach to finding the Nash equilibriums for pure and mixed strategies. For Prudential strategies, linear programs can be used for each player to find the security levels. The Nash arbitration method will be shown as a nonlinear optimization problem. We illustrate all these with MS-Excel and a Solver Macro template designed as a technology assistant.

Key words: Partial conflict games, nonzero sum games, game theory, Nash equilibrium, prudential strategies, security levels, Nash arbitration, linear programming, nonlinear programming, MS-Excel

Introduction

In our interdisciplinary Department of Defense Analysis at the Naval Postgraduate School, we teach a three course sequence in mathematical modeling for decision making. In the first course, we teach basic linear programming both using the two-variable graphical simplex technique and the Excel Solver using SimplexLP. In the third course, we teach *models of conflict* that concentrates on game theory.

In this 3rd course, we teach the basic concepts and solution techniques for game theory. In our class we use the Straffin text [8] as well as Chapter 10 from Giordano, Fox, and Horton [4]. We will not cover the basic solution techniques in this paper other than to illustrate the movement diagram.

Our students must complete a course project of their own choice using one of the modeling techniques from class. Students use the modeling process in their project: they identify the problem; they list the appropriate assumptions with justifications; they explain why their modeling technique is selected; they solve the model; interpret the solution; perform sensitivity analysis (if applicable); and they discuss strengths and weaknesses of their modeling approach. Here is short list of some of the game theory projects:

- Game Theory with US and Non-State actors.
- Game Theory in Cameroon-Nigeria dispute.
- Game Theory in PMI and US military tasks.
- COIN Game.
- The Somali Pirates game.
- US-Afghanistan drug dilemma.
- US-Afghanistan Regional Game.
- US Coin Operations Game.
- Dealing with Safe Havens as a Game.
- IEDS and Counter-IEDS as a Game.
- Game theory for Courses of Combat Actions.

In the past, our coverage did not cover much linear programming or nonlinear programming, so our solution processes were limited to two-person, two strategy games using algebraic methods because of the complexity of the solution mechanics. Recently, we have added more applications of linear programming as a solution technique so students might add more reality to the number of possible strategies available to the players.

Partial Conflict Games

Let's first define a partial conflict game. As opposed to a total conflict game where if a player wins *x* his opponent loses *x*, in a partial conflict game the players are not strictly opposed, so it is possible for both players to win or lose some value.

In a partial sum game the sum of the values for the two players do not sum to zero. For example, consider the following game where the sums of the outcomes do not all sum to zero.

		Player II	
	C_1		C_2
R_1	(2, 4)		(1, 0)
Player I			
R_2	(3, 1)		(0, 4)

In Figure 1 we note that a plot of the payoffs to each player do not lie on a line, indicating that the game is a partial conflict game because total conflict game values lie in a straight line.

What are the objectives of the players in a partial conflict game? In total conflict, each player attempts to maximize his payoffs and necessarily minimizes the other player in the process. But in a partial conflict game, a player may have any of the following objectives from Giordano et al. [6]

1. **Maximize his payoffs.** Each player chooses a strategy in an attempt to maximize his payoff. While he reasons what the other player's response will be he does not have the objective of insuring the other player gets a "fair" outcome. Instead, he "selfishly" maximizes his payoff.

2. **Find a stable outcome.** Quite often players have an interest in finding a stable outcome. *A Nash equilibrium outcome is an outcome from which neither player can unilaterally improve*, and therefore represents a stable situation. For example, we may be interested in determining whether two species in a habitat will find equilibrium and coexist, or will one species dominate and drive the other to extinction? The Nash equilibrium is named in honor of John Nash who proved [7] that every two-person game has at least one equilibrium in either pure strategies or mixed strategies.

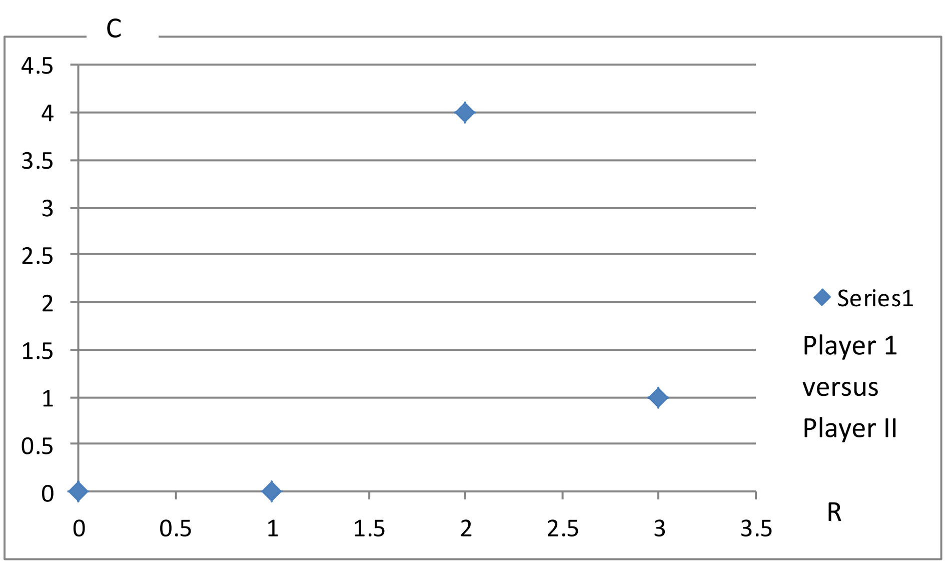

Figure 1. Payoffs in a partial conflict game do not lie on a line.

3. **Minimize the opposing player.** Suppose we have two corporations whose marketing of products interact with each other, but not in total conflict. Each may begin with the objective of maximizing his payoffs. But, if dissatisfied with the outcome, one, or both corporations, may turn hostile and choose the objective of minimizing the other player. That is, a player may forego their long-term goal of maximizing their own profits and choose the short term goal of minimizing the opposing player's profits. For example, consider a large, successful corporation attempting to bankrupt a "start-up venture" in order to drive him out of business, or perhaps motivate him to agree to an arbitrated "fair" solution.

4. **Find a "mutually fair" outcome, perhaps with the aid of an arbiter.** Both players may be dissatisfied with the current situation. Perhaps, both have a poor outcome as a result of minimizing each other. Or perhaps one has executed a "threat' as we study below, causing both players to suffer. In such cases the players may agree to abide by the decision of an arbiter who must then determine a "fair" solution. [7]

In this introduction to partial conflict games, we will assume that both players have the objective of maximizing their payoffs. Next we must determine if the game is played without communication or with communication. "Without communication" indicates that the players must choose their strategies without knowing the choice of the opposing player. For example, perhaps they choose their strategies simultaneously. The term "with communication" indicates that perhaps one player can move first and make his move known to the other player, or that the players can talk to one another before they move. We assume that our games do not allow communication and are played simultaneously.

Further we assume our players are rational, attempting to obtain their best outcomes and that games are repetitive.

One method to find a pure strategy solution is the movement diagram. We define the movement diagram as follows:

<u>Movement Diagram</u>**:** For Player one, examine the first value in the coordinate and compare R_1 to R_2. For each C_1 and C_2 draw an arrow from the smaller to larger values between R_1 and R_2. For Player two examine the second value in the coordinate and compare C_1 to C_2. For each R_1 and R_2 draw an arrow from the smaller to larger values between C_1 and C_2.

For example, under C_1, we draw an arrow from 2 to 3 and under C_2 from 0 to 1. Under R_1 we draw the arrow from 0 to 4 and under R_2 from 1 to 4. We show this in Figure 2.

		Player II	
	C_1		C_2
R_1	*(2, 4)*	←	*(1, 0)*
Player I	↓		↑
R_2	*(3, 1)*	→	*(0, 4)*

Figure 2. Movement diagram.

Using the Excel template, Figure 3, the arrows indicate "false" in all directions so there is no pure strategy.

We follow the arrows. If the arrows lead us to a value or values where no arrows points out then we have a pure strategy solution. If the arrows move in a clockwise or counter-clock wise direction then we have no pure strategy solution. Here we move counter-clock wise and have no pure strategy solution. As Nash proved all games have a solution either by pure or mixed strategies. As a matter of fact others (Barron [1]; Houseman and Gillman[5]) have shown that some partial conflict games have both a pure and mixed (equalizing) strategy.

For Solving a 2 x 2 game for Equalizing Strategies

Step 1. Enter Rose's and Colin's Values into the appropriate cells

				Colin			
			C1			C2	
	R1	2	4	<--	1	0	
Rose					/|\	2	
	R2	3	1	-->	0	4	
Follow the arrows:			FALSE				0
			FALSE				0

Figure 3. Screenshot of template for movement diagram.

We start here by defining the mixed (equalizing) strategy for a partial conflict game.

Rose's game: Rose maximizing, Colin "equalizing" is a total conflict game that yields Colin's equalizing strategy.

Colin's game: Colin maximizing, Rose "equalizing" is a total conflict game that yields Rose's equalizing strategy.

Note: If either side plays its equalizing strategy, then the other side "unilaterally" cannot improve its own situation (it stymies the other player).

We will call this strategy, an equalizing strategy. Each player is restricting what his opponent can obtain by insuring no matter what they do that his opponent always gets the identical solution (Straffin [8]).

Methods to Obtain the Equalizing Strategies

We present two methods to obtain equalizing strategies and we will apply these methods to our previous example. The two methods are: linear programming and nonlinear programming. We state here that linear programming works only because each player has only two strategies.

Linear Programming with Two Players and Two Strategies Each

This translates into two maximizing linear programming formulations as shown in Equations (1) and (2). Formulation (1) provides the Nash equalizingsolution for Colin with strategies played by Rose while formulation(2) provides the Nash equalizing solution for Rose and strategies played by Colin. The two constraints representing strategies are implicitly equal to each other per this formulation (Fox, [3]).

Maximize V

Subject to:

$$N_{1,1}x_1 + N_{2,1}x_2 - V \geq 0$$
$$N_{1,2}x_1 + N_{2,2}x_2 - V \geq 0$$
$$(N_{1,1} - N_{1,2})x_1 + (N_{2,1} - N_{2,2})x_2 = 0 \qquad (1)$$
$$x_1 + x_2 = 1$$

Nonnegativity

Maximize v
Subject to:

$$M_{1,1}y_1 + M_{1,2}y_2 - v \geq 0$$
$$M_{2,1}y_1 + M_{2,2}y_2 - v \geq 0$$
$$(M_{1,1} - M_{2,1})y_1 + (M_{1,2} - M_{2,2})y_2 = 0 \qquad (2)$$
$$y_1 + y_2 = 1$$

Nonnegativity

With our example, we obtain the following formulation

Maximize V
Subject to:

$$4x_1 + x_2 - V \geq 0$$
$$0x_1 + 4x_2 - V \geq 0$$
$$4x_1 - 3x_2 = 0$$
$$x_1 + x_2 = 1$$

Nonnegativity

and

Maximize v
Subject to:

$$2y_1 + y_2 - v \geq 0$$
$$3y_1 + 0y_2 - v \geq 0$$
$$-y_1 + y_2 = 0$$
$$y_1 + y_2 = 1$$

Nonnegativity

The solution, via the Excel's solver, is:

	Linear Programming						
Decsion Variables							
x1	0.571429						
x2	0.428571						
vc	1.714286						
y1	0.5						
y2	0.5						
vr	1.5						
OBJ	3.214286						
Constraints			0	0		*2y1+y2-vr>=0*	
			0	0		*3*y1-vc>=0*	
			1	1		*y1+y2=1*	
			1	0		*4x1+x2-vc>=0*	
			0	0		*4x2-vc>=0*	
			1	1		*x1+x2=1*	
			0	0		*-y1+y2=0'*	
			0	0		*3x1-4x2=0*	

3/7 x_1, *4/7* x_2 corresponding to *3/7* R_1, *4/7* R_2 *and* ½ y_1, ½ y_2 corresponding to ½ C_1, ½ C_2. The Nash equilibrium is (3/2, 16/7).

Nonlinear Programming Approach for Two or More Strategies for Each Player

For games with two players and more than two strategies each, we present the nonlinear optimization approach by Barron [1]. Consider a two person game with a payoff matrix as before. Let's separate the payoff matrix into two matrices **M** and **N** for players I and II. We solve the following nonlinear optimization formulation in expanded form, in Equation (3).

$$Maximiz \sum_{i=1}^{n} \sum_{j=1}^{m} x_i a_{ij} y_j + \sum_{i=1}^{n} \sum_{j=1}^{m} x_i b_{ij} y_j + - p - q$$

Subject to

$$\sum_{j=1}^{m} a_{ij} y_j \leq p, \quad i = 1,2,\dots,n,$$
$$\sum_{i=1}^{n} x_i b_{ij} \leq q, \quad j = 1,2,\dots,m, \qquad (3)$$
$$\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j = 1$$
$$x_i \geq 0, y_j \geq 0$$

We return to our previous example. We define *M* and *N* as:

$$\boldsymbol{M} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{0} \end{bmatrix} \; \boldsymbol{and} \; \boldsymbol{N} = \begin{bmatrix} \mathbf{4} & \mathbf{0} \\ \mathbf{1} & \mathbf{4} \end{bmatrix}$$

We define x_1, x_2, y_1, y_2 as the probabilities for players playing their respective strategies.

By substitution and simplification, we obtain

Maximize $6 y_1 x_1 + 4 y_1 x_2 + x_1 y_2 + 4 x_2 y_2 - p - q$

Subject to:

$x_1 + x_2 = 1$
$y_1 + y_2 = 1$
$4x_2 - q \leq 0$
$4x_1 + x_2 - q \leq 0$
$2y_1 + y_2 - p \leq 0$
$3y_1 - p \leq 0$
Nonnegativity

A		
	2	1
	3	0
B	4	0
	1	4
dv		
x1	0.428572	
x2	0.571429	
p	1.500001	
y1	0.5	
y2	0.5	
q	2.285715	

We find the exact same solution as before with the larger screenshot.

Finding a Solution

According to Straffin [8], a Nash equilibrium is a solution if and only if it is unique and Pareto Optimal. Pareto optimal refers to the northeast region of a payoff polygon where the payoff polygon is found as the convex set formed by the outcome coordinates, Figure 4.

A	B	C	D	E	F	G
Decision Variables						
x1	0.428571				Objective Function	
x2	0.571429				z	0
x3	0					
x4	0					
x5	0			X^TAY^T	1.5	
x6	0			X^TBY^T	2.285714	
x7	0					
x8	0					
x9	0					
x10	0					
P	1.5					
v2	0					
y1	0.5					
y2	0.5					
y3	0					
y4	0					
y5	0					
y6	0					
y7	0					
y8	0					
y9	0					
y10	0					
q	2.285714					
v4	0					

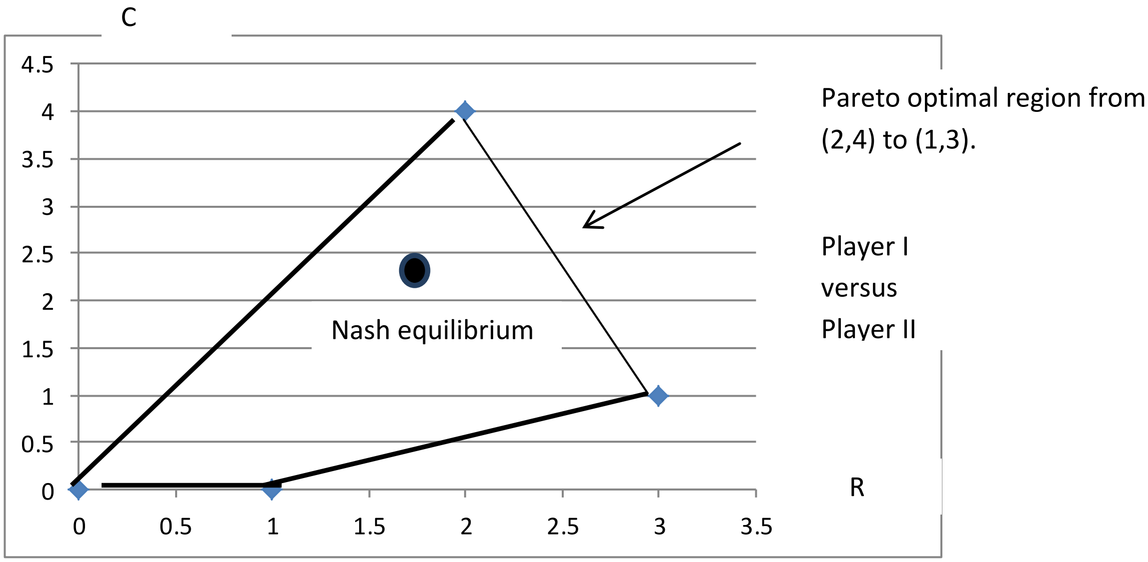

Figure 4. Payoff polygon and Pareto optimal region.

We see in the figure that the Nash equilibrium (1.5, 2.28) is not Pareto optimal and not the solution that we should seek.

At this point, we might try to allow communication and try strategic moves which we do not describe here but can be reviewed in Giordano, et al. [6]. Further, we might want to show the method of Nash arbitration although we do not illustrate that here.

Conclusions

We have shown how to use optimization to solve the Partial Conflict games. We point out that we built many Excel templates to assist with finding these results for the Partial Conflict games. The author will provide these templates or detailed instructions upon request.

Dedication

This paper is dedicated in the memory of John Forbes Nash, Jr., whose influence during a class visit at NPS in 2009, had far reaching effects on my knowledge of game theory.

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Biographical Information

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