

FACILITATING MATHEMATICAL MODELING AND INQUIRY THROUGH POPULATION GROWTH AND FALLING BODY PROBLEMS

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Abstract

The Naval Postgraduate School's Department of Defense Analysis improves student analytic skills by enabling them to make informed, rational decisions. Among the means of attaining this end, we emphasize the connection between mathematics and its application in real problems. In addition, the use of technology (Excel spreadsheets, in this case) facilitates inquiry and experimentation as part of the modeling and problem solving process. This paper addresses a series of exercises that combine mathematical modeling, problem-solving, inquiry, experimentation, and Excel computer simulations. The exercise begins with the idea of proportionality, applied immediately in a discrete model for population growth. This concept is expanded and modified to create a discrete logistic growth model. Students apply this updated population growth model to estimate a model parameter for given population data. Next, the students are introduced to a discrete model for the position, velocity, and acceleration of a falling body. Then students are asked to modify the model to incorporate drag that is assumed first to be proportional to the object's velocity and secondly to the square of the object's velocity, and apply their model to given data. Finally, for each assumption, students are asked to determine the proportionality constant as well as the object's terminal velocity.

Introduction

Students matriculating through the Defense Analysis (DA) curriculum at the Naval Postgraduate School (NPS) come from a diverse background. Most are mid-level military officers from all U.S. services, but we also meet civilians from the federal government, as well as international officers. One thing most students have in common is a limited mathematics background and a requirement to make informed decisions. The DA curriculum includes a three course mathematics sequence featuring mathematical modeling, probability and statistics, and game theory. The intent of the sequence is to make the students better informed, analytic decision makers. The prerequisite for this three course sequence is college algebra.

Our first course covers deterministic modeling, including topics such as graphs as models, dynamical systems models (implemented using Excel spreadsheets), concepts of agent-based modeling, model fitting with least squares, and an introduction to linear programming models in two-variables (graphical) and multi-variable (implemented using Excel solver). Our second course introduces stochastic modeling. In this course, we review descriptive statistics and classical probability, including Bayes' Theorem, basic distributions (binomial, Poisson, uniform, exponential, and normal) and their uses in modeling and reliability analysis, and Monte Carlo simulation modeling. Our third course, Models of Conflict, covers a review of expected

value for use in decision trees, linear programming as a decision tool, and game theory.

The span of mathematical abilities found among our students reflects national trends, and so a good portion of our students exhibit deficiencies in basic algebra skills. College algebra constitutes the only prerequisite for our courses, but many of our students have not seen the material for as little as seven and as long as twenty years. To our advantage, the students are nearly uniformly motivated to learn, and most quickly re-acquire the necessary algebraic proficiency within a few lessons. Excel is the technology platform used in the sequence (since it is readily available on most computers), and it helps to ease some of the quantitative burdens that students might face. In addition, Excel facilitates discovery, inquiry, and model implementation. Many students are already familiar with its use; those uninitiated are introduced to it and its capabilities through a series of in-class exercises. By the end of the first course, the typical student is proficient at using Excel for implementing mathematical models and making informed decisions from them.

This paper describes a sequence of exercises from our first course. The objective of the exercises is to begin the process of developing student mathematical modeling skills.

Proportionality

We begin the course with a review of some mathematical basics, including numbers, operations, and especially the equation of a line passing between two points. In this review of basic mathematics, we define proportionality by stating that “some entity p is proportional to another entity q implies that

$$p = k*q \quad (1)$$

for some constant k .” Having already reviewed the equation of a line, the students understand that equation (1) is simply a line through the origin in the q - p plane with slope k .

A simple, discrete growth model

Next we develop a model for exponential growth, in discrete form, by applying the definition of proportionality. The claim is that the rate of change in a population, from one time period to the next, is proportional to its current size. Defining $p(n)$ to be the population size at time n , then by the definition of proportionality we have

$$p(n+1) - p(n) = \mathbf{r} * p(n) \quad (2)$$

where the left hand side of this equation is the rate of change in the population from time period n to time period $n+1$, and \mathbf{r} is the proportionality constant (in bold font to draw student attention to form and role).

Writing the difference equation (2) as

$$p(n+1) = p(n) + \mathbf{r} * p(n) \quad (3)$$

and defining the initial population as $p(0) = p_0$, we observe that the population would grow exponentially and without bound. In this form, we might consider that \mathbf{r} is the growth rate.

During class, students apply this model to the growth of money in an account, where \mathbf{r} would be the interest rate per time period, and n would represent time.

Exercise 1

Assume the initial population size is 2, that is, $p(0) = 2$. Find the population size after 50 time intervals, $p(50)$, for various positive values of \mathbf{r} . Make a conjecture about the behavior of $p(n)$ for large n and positive \mathbf{r} values.

Exercise 2

What process(es) on the Earth grow without bound?

Exercise 3

For the data in following table, estimate the growth rate, \mathbf{r} .

Table 1. Time and population data.

time, n	p(n)
0	2
1	2.14746
2	2.640809
3	2.762708
4	3.325214
5	3.719376
6	4.203125
7	4.269002
8	4.864998
9	5.227523
10	5.56491
11	6.69713
12	7.790299
13	8.152035
14	9.555009
15	11.12947
16	13.25046
17	15.69848
18	19.37997
19	19.8353
20	22.13117

Exercise 4

How many years would it take an initial deposit of \$1000 to double in value for an annual interest rate of 4% (assuming no other deposits or withdrawals)?

Exercise 5

The “Rule of 72” claims that it takes n years at $r\%$ annual interest for an amount to double, provided that $n * r = 72$. How accurate is this model? Make a table of the absolute and relative error in the “Rule of 72” model for various n when r takes on integer values between 1% and 15%.

A discrete, logistic growth model

Recognizing that environmental conditions and competition serve to limit growth by establishing a carrying capacity, we modify our growth model to incorporate a variable growth rate. We might expect for small population sizes that growth might be somewhat akin to exponential growth, but that the population growth would approach zero when the population approaches its environmental carrying capacity, M . A model with these properties might look like this:

$$p(n + 1) = p(n) + \mathbf{k} * (\mathbf{1} + \mathbf{p(n)/M}) * p(n) \tag{4}$$

This is a discrete form of the logistic growth model. The bold text is meant to draw student attention to the form of the model: though this updated model differs from the earlier, simpler model, the bold-font term’s place in the equation is meant to connote its role in model behavior; in this case, rather than a constant growth rate, in (4) the rate varies with the population size, $p(n)$.

Exercise 6

Assume the initial population size is 2, $p(0) = 2$, and the carrying capacity is 100, $M=100$. Find the population size after 50 time intervals, $p(50)$, for various values of k .

Exercise 7

Let $p(0) = 150$, let $M=100$, and find $p(50)$ for various values of k .

Exercise 8

Data for yeast biomass as a function of time (hours) is provided in the table, below. Assume that the yeast biomass is limited to 665 units and that it follows a discrete logistic growth model. Estimate the growth parameter, k .

Table 2. Time and Yeast biomass data

Time (hrs)	Observed Yeast Biomass p(n)
0	11.2
1	18.3
2	29
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3
8	350.7
9	441
10	513.3
11	559.7
12	594.8
13	629.4
14	640.8
15	651.1
16	655.9
17	659.6
18	661.8

Summary

Once the students have completed the exercises, they should have developed an understanding of exponential growth, logarithmic growth, and the role of the growth rate in the models' behavior. With that understanding, they should be ready to undertake a somewhat more involved model.

Falling body problem, with and without drag

Galileo's novel use of an accurate time piece allowed him to record a falling object's position at known times. He then computed the differences between positions, which yielded average velocity over the time period. From the average velocities, he again computed differences which yielded average accelerations. His findings: the accelerations (the second differences in position) were essentially constant. It turns out that objects fall at a constant acceleration provided that neither friction nor buoyancy forces act upon them. [Several hundred years later, an Apollo astronaut verified Galileo's model by simultaneously releasing a hammer and a feather on the lunar surface – they reached the surface at the same time.] However, when an object falls in the Earth's atmosphere, friction and buoyancy forces result in non-constant acceleration.

The following table contains position (height) data recorded as a relatively massive object was released to fall through the atmosphere. Its position is recorded every 0.5 seconds and is depicted in the second column called "Falling body height (feet)". The third column contains computed positions (heights) based on a model for the position of a falling object without drag or buoyancy forces. We will derive that model, below.

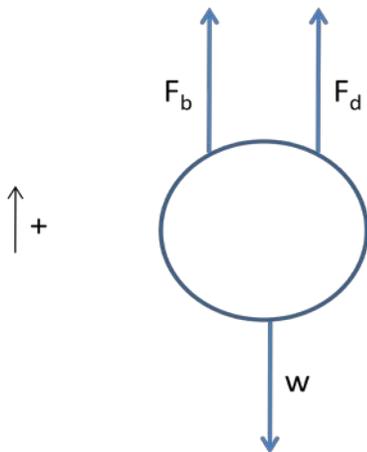
Table 3. Time, falling body data, and falling body model (without drag) calculated heights.

<u>Time (sec)</u>	<u>Falling body data (feet)</u>	<u>Model (no drag) heights</u>
0	200	200
0.5	195	195
1	186	182
1.5	172	161
2	153	132
2.5	131	95
3	107	50
3.5	82	-3
4	44	-64

Look at the Table 3 and note that the positions in the third column decrease (fall) more quickly than the recorded data in column two. This should make sense: drag acts to retard the downward motion of the object, making the object's position at each time higher than the no drag position. We also note that the first two positions are the same for both the falling body height and the no drag model (columns 2 and 3 at times $t = 0$ and $t = 0.5$, above). The reason for the first two row entries being the same is due to the structure of the no drag model (a second-order difference equation), which necessitates two known positions from which to start. We will derive the equation for the object's position in the no drag, no buoyancy case, and it will be clear why the two positions are needed. Also, we will use the model to obtain (verify) the cell entries for column 3.

Those who have had a course in physics may recall the importance of a free body diagram. We will use one to depict the forces acting upon a falling object.

Here the circle represents the object, F_b represents the upward acting buoyancy force, F_d represents the upward acting drag force, and w represents the object's downward acting weight. The arrow with the plus sign indicates that "up"



is positive. For a relatively massive object falling in the atmosphere, like the object in our problem, we will take the buoyancy force to be essentially zero.

For now, let us assume that there is no drag, and let $h(n)$ represent the object's height at time period n . Let $v(n)$ represent the velocity at time period n , and let $a(n)$ represent the acceleration at time period n . The acceleration at time n is given by the difference between the velocities at times $n+1$ and n :

$$a(n) = v(n+1) - v(n). \quad (5)$$

Similarly, the velocity at time n is given by the difference in the position at times $n+1$ and n :

$$v(n) = h(n+1) - h(n). \quad (6)$$

From (6), it follows that

$$v(n+1) = h(n+2) - h(n+1), \quad (7)$$

which represents the velocity at time $n+1$ expressed in terms of the associated positions. Using the two equations for velocity, (6) and (7), we obtain an expression for the acceleration in terms of the positions by substituting them into (5):

$$a(n) = v(n+1) - v(n) = [h(n+2) - h(n+1)] - [h(n+1) - h(n)] \quad (8)$$

yielding

$$a(n) = h(n+2) - 2h(n+1) + h(n). \quad (9)$$

Since the only force on the object is the propulsive force of its weight, where weight = $m \cdot g$; and since 'up' is positive, then this downward force will require a negative sign. Thus, the propulsive force is given by

$$w = -m \cdot g. \quad (10)$$

Since the sum of the forces is equal to mass time acceleration (Newton's Second Law), then for the no-drag case

$$-m \cdot g = m \cdot a(n), \quad (11)$$

since there is only one force, the downward acting weight. Dividing by the object's mass, m , and rearranging we obtain:

$$a(n) = -g. \quad (12)$$

Substituting (9) into (12) yields

$$h(n+2) - 2 \cdot h(n+1) + h(n) = -g. \quad (13)$$

Solving for $h(n+2)$, we obtain:

$$h(n+2) = 2 \cdot h(n+1) - h(n) - g. \quad (14)$$

Equation (14) is the difference equation that is implemented in column 3, above, for time $t = 1$ through $t = 4$, which corresponds to $n = 2$ through $n = 8$.

Note that we have to be given two positions, $h(0) = 200$ and $h(1) = 195$, in order to compute $h(2)$. Thus, $h(2) = 2 \cdot h(1) - h(0) - g$. If $g = 8$, then $h(2) = 2 \cdot (195) - 200 - 8 = 182$. (Why is the magnitude of the acceleration a constant 8, rather than 32? Hint: what is the length of the time period in the given data?)

Exercise 9

Implement the no drag model in Excel, and obtain a match to column 3 in Table 3.

Exercise 10

Assume that drag force is proportional to velocity. (Think: in which direction does this force act? In which direction is the velocity? Therefore, be careful with the +/- sign.) Use Newton's Second Law to obtain a model for the position of the falling object under the assumption that the drag force is proportional to the object's velocity. Then incorporate the model into a spreadsheet, and obtain an estimate for the value of the proportionality constant. How good is this model? What criterion did you use to judge its adequacy?

Exercise 11

Assume that the drag force is proportional to the velocity squared. (Again, think: in which direction does this force act? In which direction is the velocity? Therefore, be careful with the +/- sign.) Use Newton's Second Law to obtain a model for the position of the falling object under the assumption that the drag force is proportional to the object's velocity squared. Then incorporate the model into a spreadsheet, and obtain an estimate for the value of the proportionality constant. How good is this model? What criterion did you use to judge its adequacy?

Exercise 12

If the object were to continue falling (perhaps past $h = 0$), does your model in Exercise 10 and/or Exercise 11 lead to a terminal velocity? If so, then what is it, and how long does a falling object take to attain terminal velocity? Does the terminal velocity depend on your proportionality constant? If so, then how sensitive is the terminal velocity to the value of the proportionality constant?

Exercise 13

Might the force due to drag be proportional to the velocity raised to some exponent other than 1 or 2? If so, then try that model, and comment on its adequacy relative to the other models for drag.

Conclusion

The modeling and analysis supported by these problems are conducive to developing critical thinking, an objective of the three course math sequence. The implementation of the models into Excel helps to ease some of the quantitative burdens, and it facilitates discovery and inquiry. The problems described here are substantial aids in helping students to become proficient at using Excel for implementing mathematical models and making informed decisions from them.

Biographical Information

Dr. Michael Jaye received his B.S. degree from the United States Military Academy at West Point, NY, and his M.S. and Ph.D. degrees from Rensselaer Polytechnic Institute, Troy, NY. He is an associate professor at the Naval Postgraduate School. He serves as a judge for both the Collegiate Mathematical Contest in Modeling (MCM) and for the High School Mathematical Contest in Modeling (HiMCM). His interests include applied mathematics, modeling, differential equations, agent based modeling, and computer simulations of human social and behavioral interactions.

Dr. Robert E. Burks graduated from the United States Military Academy (USMA) in 1987 with a B.S. in Aerospace Engineering. He later received a M.S. in Operations Research from the Florida Institute of Technology and a Ph.D. in Operations Research from the Air Force Institute of Technology. He served as an Assistant Professor and Director for the Developmental Mathematics Program in the Department of Mathematical Sciences at the USMA and is currently serving as an Associate Dean for the Graduate School of Operational and Information Sciences and an Assistant Professor in the Department of Operations Research at the Naval Postgraduate School. His research interests include mathematical modeling, agent based modeling, and the mathematics of 3D animation and modeling.

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