VISUALIZING LINEAR TRANSFORMATIONS USING THE CONDITION NUMBER OF A MATRIX

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Abstract

Linear transformations mapping R^3 to R^3 will be considered. The computer algebra system Maple will be used to demonstrate what happens to a vector in R^3 when it is transformed by the linear transformation. A procedure was created in Maple to graph both the unit sphere and the figure that results when vectors of length one are multiplied by a 3 x 3 matrix. The resulting figures are used to visualize important concepts in linear algebra. The condition number of a matrix is introduced to explain potentially confusing figures.

When students think of linear transformations, they usually think in terms of matrices and their corresponding pivots, rows, and columns. The best way for students to understand linear transformations is by having them actually see what happens when vectors are multiplied by matrices. The method presented in this paper allows students to comprehend what is really happening to vectors when they are transformed. While other software packages allow the presentation of similar results, the use of Maple encourages the advanced student to create procedures that illustrate other mathematical properties.

In this paper, linear transformations mapping R^3 to R^3 will be considered. Maple will be used to demonstrate what happens to a vector in R^3 when it is transformed by the linear transformation T. Certainly it not possible to demonstrate what happens to every three dimensional vector under a linear transformation. However, that is unnecessary. It is sufficient to demonstrate what happens to vectors of length one. Suppose that ||v|| represents the length of the nonzero vector v. Then v/||v|| is a vector of length one and T(v) = T(||v||*v/||v||). But since T is linear, T(v) = ||v||*T(v/||v||). So if the student knows what T(v/||v||) looks like, that student also knows that T(v) is just a positive constant multiple of it. Thus, T(v) is simply a stretching or shrinking of T(v/||v||).

Every transformation from R^3 to R^3 can be represented by a 3x3 matrix. A procedure was created in Maple to graph both the unit sphere and the figure that results when vectors of length one are multiplied by a 3 x 3 matrix. Since the transformation is linear, the 0 vector maps to the 0 vector. Since the vector 0 is not on the unit sphere, the fact that T(0)=0 will not be represented. For every other nonzero vector, division by ||v|| puts v/||v|| on the unit sphere. The transformation of this corresponding unit vector will be represented on the resulting graph. Both the LinearAlgebra and the plots libraries are used in the following code. The procedure that creates the two figures follows.

```
> sphere:=proc(A);
>
P1:=Matrix([[sin(x)*cos(y)],[sin
(x)*sin(y)],[cos(x)]]):
>
P2:=MatrixMatrixMultiply(A,P1):
>
P1:=convert(convert(P1,vector),l
ist):
>
P2:=convert(convert(P2,vector),l
ist):
>
g1:=plot3d(P1,x=0..Pi,y=0..2*Pi,
style=PATCH,color=black,scaling=
CONSTRAINED):
>
g2:=plot3d(P2,x=0..Pi,y=0..2*Pi,
style=WIREFRAME,color=black,thic
```

```
kness=1,axes=normal,scaling=CONS
TRAINED);
> display3d({g1,g2});
> end:
```

Consider a transformation T(x, y, z) = (4x + y + z, x + 3y + z, x + y + 2z). The matrix associated with this transformation is:

```
> 
A:=Matrix([[4,1,1],[1,3,1],[1,1,
2]]);
```

$$A := \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Consider what happens when the points on the unit sphere are multiplied by this matrix. In each of the following figures, the unit sphere is represented by the black sphere and the transformed sphere is represented by the second shape.

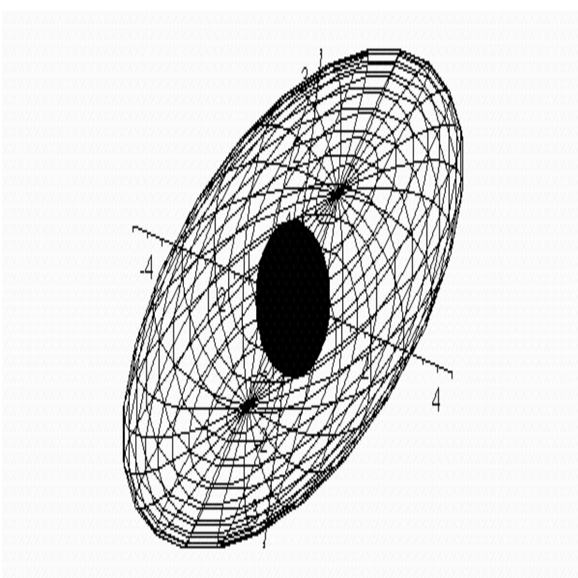

Figure 1.

Students see that the sphere has been transformed into an ellipsoid. Consider this linear transformation. The students can rotate the above graph to analyze it from all different viewpoints. Students very quickly begin to make correct conjectures about the way that such a transformation behaves. For example, no vectors on the unit sphere get mapped to the zero vector. Since the nullspace consists of just the zero vector, the nullity must be 0. The range of the transformation is a space with dimension three. Thus, the rank is three. Each point on the ellipsoid comes from only one point on the sphere. The mapping is one-to-one and it maps R^3 onto R^3. Therefore, the matrix must have an inverse.

Now consider the linear transformation T(x, y, z) = (2x + y + 2z, 3x + 3y + 3z, x + 2y + z). The matrix associated with this transformation is:

```
> 
B:=Matrix([[2,1,2],[3,3,3],[1,2,
1]]);
```

$$B := \begin{bmatrix} 2 & 1 & 2 \\ 3 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Consider what happens when the points on the unit sphere are multiplied by this matrix.

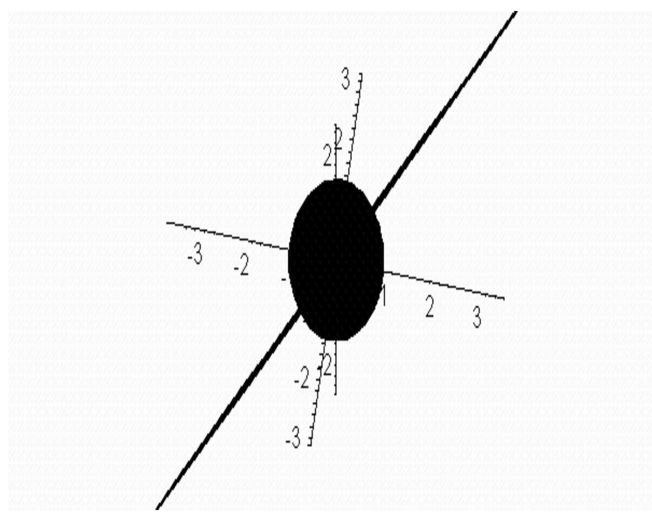

Figure 2.

Different views of the resulting graph are necessary in order to understand what is happening here. In Figure II, it looks as though the sphere has been transformed into a line. However, a different view in Figure III shows that the sphere has really been transformed into an ellipse and its interior.

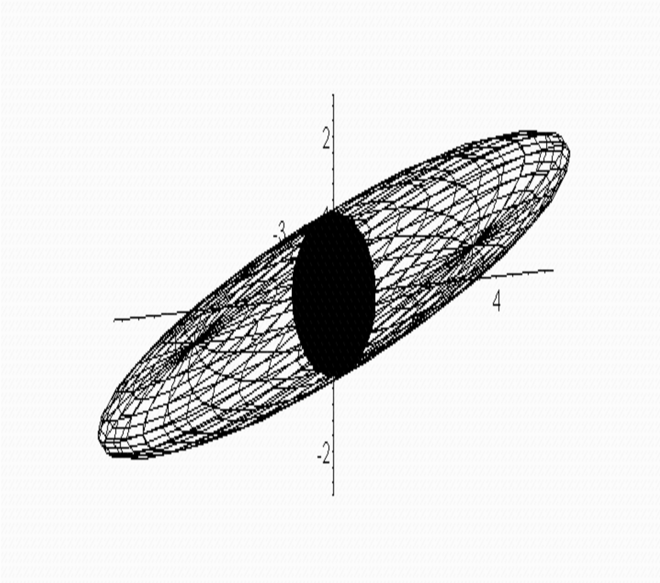

Figure 3.

Consider this second linear transformation. By analyzing the above figures, students can see that the rank of the matrix is the same as the dimension of the range. They are looking at a figure with dimension two when they look at the range. Since one dimension was lost during the transformation, the nullspace cannot include only the zero vector. Students see that nonzero vectors have mapped to the zero vector. The dimension of the nullspace has to be one since the dimension of the range is two.

Now consider a transformation T(x, y, z) = (x + y + z, 2x + 2y + 2z, 1/2 x +1/2y + 1/2 z). The matrix associated with this transformation is:

```
> C:=Matrix([[1,1,1],[2,2,2],[1/2, 1/2,1/2]]);
```

$$C := \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Consider what happens when the points on the unit sphere are multiplied by this matrix.

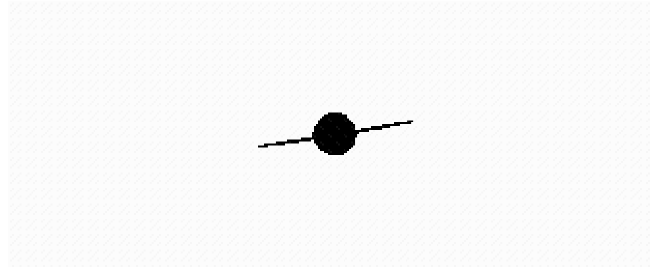

Figure 4.

Consider this third linear transformation. Different views of the above figure yield virtually the same figure. By analyzing it, students can see that the rank of the matrix is one and that the dimension of the nullspace is two. The theorem stating that the dimension of the original space equals the dimension of the range plus the dimension of the nullspace finally makes sense. They realize that such a transformation cannot be invertible, and can see the relationship between the concept of a one-to-one mapping and the invertibility of the transformation.

Unfortunately, there can be problems with such an analysis. Some nonsingular transformations actually look singular when using this geometric approach. This occurs when some of the unit vectors are sent a large distance away from the sphere while others are sent a very small distance away. The result is a transformed sphere that looks like a plane or a line but that is actually a long, thin ellipsoid. As a result, an analysis of the graph associated with a particular linear transformation is not complete without an understanding of the condition number of the corresponding matrix.

The condition number of a matrix measures the relationship between the maximum stretching and the maximum shrinking (or the minimum stretching if no shrinking occurs) of the vectors on the unit sphere. The maximum stretching is the distance from the origin to the point on the ellipsoid farthest away. The maximum shrinking is the distance from the origin to the point on the ellipsoid which is closest. The condition number is the quotient of the maximum stretching and the maximum shrinking. If the maximum stretching is large and the maximum shrinking is small, the ellipsoid will be long and thin. The result will be a very large condition number. If the matrix is singular, some points on the unit sphere will be mapped to zero. As a result, maximum shrinking will be 0 and the condition number will be undefined.

The condition number also measures the sensitivity of the matrix to small changes in its entries. That is, it reflects the degree to which the solution of a system of equations is altered by small changes in the matrix. A large condition number corresponds to a matrix which is sensitive to small changes. Vectors that are close together can get mapped to vectors that are very far apart. Such a matrix is called ill-conditioned. A small condition number corresponds to a matrix which is not sensitive to small changes. Vectors that are close together are mapped to vectors that are relatively close. Such a matrix is called well-conditioned. Graphically, an ill-conditioned matrix transforms the sphere into an ellipsoid, however, the ellipsoid is so thin that from one particular view, it looks more like a line. An understanding of the concept of condition number is very important in order that students understand what is happening in the following example. Note that the vector [1, 1, 1] is a solution of the following system of equations.

$$x + 2y + z = 4$$
$$2x + 4.0001\,y + 2.002\,z = 8.0021$$
$$x + 2.002\,y + 2.004\,z = 5.006$$

Suppose that slight round off error occurs in some measurements and the following system is used in place of the above system.

$$x + 2y + z = 4$$
$$2x + 4.0001\,y + 2.002\,z = 8.002$$
$$x + 2.002\,y + 2.004\,z = 5.006$$

It would seem that with such small changes, the new solution will still be close to the previous solution of [1, 1, 1]. However, the approximate solution of the second system of equations is [3.085, -0.044, 1.002]. The reason for this discrepancy is found in the fact that the condition number of the coefficient matrix associated with this system of equations is 499558.1867. Such a condition number is huge. Consider the graph of the unit sphere and the transformed sphere.

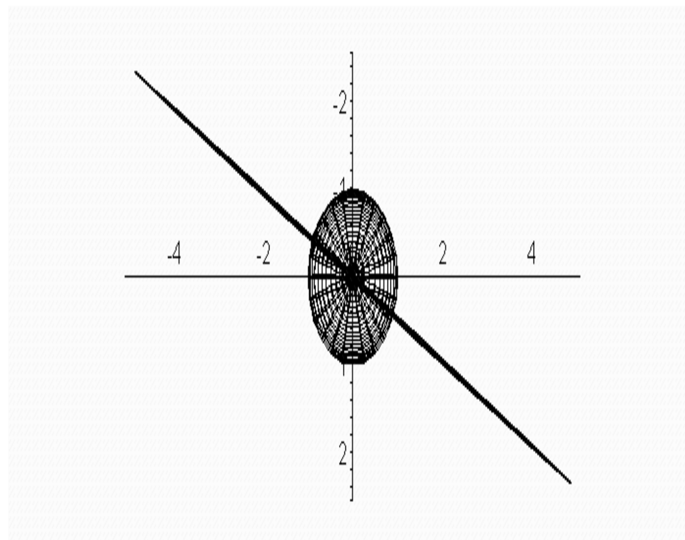

Figure 5.

While the coefficient matrix of this transformation has an inverse, Figure V reveals a graph that looks as though a singular matrix has been used. Even though no dimensions have been lost, vectors close together have been transformed to vectors so far apart that even a slight change in the values of b result in drastic changes in the solutions of the matrix equation Ax=b. The coefficient matrix is an ill-conditioned matrix. It acts more like a singular matrix than a nonsingular matrix. The graph reveals why this is true.

Classroom Use

Effective labs can be created using the graphing capabilities of Maple. An example of part of a very success lab follows. Before the lab is assigned in class, students would have been sent a copy of the lab through the classroom management system Blackboard. This would include the procedure “sphere” detailed above and at this point in the class, they understand that they must enter the procedure and load the appropriate libraries.

Lab #10: Visualizing Linear Transformations

1. Let A be the given matrix.

$$A := \begin{bmatrix} 3 & 2 & 4 \\ 4 & 5 & 3 \\ 6 & 5 & 4 \end{bmatrix}$$

a. What is a basis for the column space of A?
b. What is the range of the linear transformation associated with A?
c. What is a basis for the nullspace of A?
d. What is the nullspace?
e. What is the rank?
f. What is the nullity?

Note: To actually see the unit sphere and the transformed sphere, the code is: sphere(A);

g. What does the transformed sphere look like? How many dimensions are there in the transformed sphere? Explain in terms of rank and nullity.

2. Let B be the given matrix.

$$B := \begin{bmatrix} 5 & 4 & 3 \\ 4 & 3 & 5 \\ 4 & 3 & 5 \end{bmatrix}$$

Answer a - g above.

h. Now find the eigenvectors and eigenvalues. The code is: eigenvects(B);
i. Explain what happens when you have an eigenvalue of 0.

3. Now consider matrix C. PLEASE DO THESE QUESTIONS IN ORDER.

$$C := \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4.0001 & 2.002 \\ 1 & 2.002 & 2.004 \end{bmatrix}$$

a. Plot the sphere and transformed sphere.
b. Before you put the matrix in row reduced eschelon form, and judging just from the plot, do you think that you lost a dimension? That is, does the resulting plot look more like that of matrix A or B?
c. Now find the row reduced eschelon form of the matrix. Are you surprised?
d. Now find the eigenvalues and corresponding eigenvectors. Explain the plot using the eigenvalues.

Student reactions

The overwhelming reaction to this lab was one of surprise. As a whole, the students were positive that the row reduced eschelon form of matrix C would reveal one row of zeros. When this was not the case, many students thought that they must have made a mistake. The vast majority of students were interested in why the graph looked the way it did. They were able to explain the resulting graph when they found that one eigenvalue was very small compared to the other eigenvalues. They really liked the fact that they had been surprised. They liked it even more when they could easily figure out why this happened. In general, students who like using a computer algebra system in class are also the students who feel that it is introduced satisfactorily and is interwoven with the course. Successful labs typically allow the student to see a connection between what is being done in the lab and the material covered on tests. While using a computer algebra system, most students are empowered by their ability to tackle harder problems, visualize something interesting, and be freed from tedious calculations.

Conclusion

The computer algebra system Maple is being used effectively in the linear algebra class. This paper demonstrated how Maple can be used to help students better understand the fundamental concept of linear transformations. A graphical approach crystallizes many of the ideas that are fundamental to an understanding of linear algebra. It does not relegate linear transformations to a list of supplementary topics covered if there is time left at the end of the semester. It allows matrix theory to be seen as a tool necessary to understand linear transformations. Students exposed to this graphical presentation rarely speak about linear transformations in terms of the corresponding matrix. Rather, they speak in terms of the what shape the transformed sphere has taken. Very simply, they understand what the word "transformation" means because they have seen what a transformation does.

References

1. H. Anton & C. Rorres, *Elementary Linear Algebra: Applications Version*, 10th ed., Wiley, New York, 2009.

2. D. C. Lay, *Linear Algebra and Its Applications*, 3rd ed., Addison Wesley, New York, 2006.

3. S. J. Leon, *Linear Algebra With Applications*, 8th ed., Prentice Hall, New Jersey, 2010.

Biographical Information

Dr. Rosemary Carroll Farley is an Associate Professor of Mathematics and Computer Science at Manhattan College. Her research interest concerns the appropriate use of technology in the mathematics classroom. This includes analyzing the impact that the incorporation of a computer algebra system has in higher level collegiate mathematics courses and graphing calculator technology in courses for prospective mathematics teachers.