ALTERNATIVE COMPUTER SOLUTION METHODS FOR FREE AND FORCED VIBRATIONS

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Introduction

The dynamics course at Webb Institute includes a brief introduction to mechanical vibrations. Throughout the course we make extensive use of Mathcad software, and this usage is particularly relevant in the vibrations segment. This paper illustrates two different Mathcad approaches to solving vibration problems. In the first method, the general solution to the problem is derived by traditional methods, and Mathcad is used to apply the initial conditions and evaluate the constants in that solution. In the second method, Mathcad's ordinary differential equation solver is used to obtain the solution directly from the differential equation. There may be a vigorous philosophical debate about whether to allow students to use the computer at all in dynamics and another debate about which Mathcad-based approach is better pedagogy. In our dynamics course students have the option to use traditional or computer-based methods on homework and on tests. If they choose computer-based methods, they are permitted to use either of the Mathcad approaches discussed below.

Undamped Free Vibration

The problem statement reads: *A body with mass 25 lb_m is suspended from a spring with k = 160 lb_f/ft. At time zero, it is 0.1 feet below the static equilibrium position with a downward velocity of 2.0 ft/sec.* Students are required to find the position, velocity, and acceleration of the body as a function of time. For purposes of this paper, only the position part of the solution is presented.

Method 1: Mathcad Applies the Initial Conditions

The differential equation that governs the motion is

$$\ddot{x} + \frac{k}{m} \cdot x,$$

and the solution is

$$x = C \cdot sin\left(\sqrt{\frac{k}{m}} \cdot t + \gamma\right),$$

Differentiation of the position equation with respect to time yields an equation for velocity. Initial conditions are specified for position and velocity. The constants C and Ψ are evaluated by substituting the initial conditions into the position and velocity equations and solving the resulting equations simultaneously. This may be done symbolically or numerically by means of Mathcad. When it is done symbolically, the results are:

$$C = \sqrt{x_0^2 + \frac{\dot{x}_0^2 \cdot m}{k}} \quad and \quad \Upsilon = \arctan\left(\frac{x_0}{\dot{x}_0} \cdot \sqrt{\frac{k}{m}}\right).$$

Parts (a) and (b) of Figure 1 below illustrate the computer-aided numerical approach. The result is an <u>explicit</u> expression for position as a function of time. Velocity and acceleration may be obtained by differentiation of this expression. The graph at the bottom of Figure 1(b) shows the first second of the motion. Similar graphs for velocity and acceleration are easily obtained. In the interest of brevity, they are not included here.

Method 2: Mathcad Solves the Differential Equation

Parts (a) and (b) of Figure 2 show the Mathcad worksheet that uses the Odesolve command to produce the solution in the form of a graph. A strong cautionary note is in order here. Normally Mathcad handles units almost effortlessly. Values may be entered in any system of units, and different parameters may be entered in different systems of units. These statements do not apply to the Odesolve command. When this command is used, all parameters must be pure numbers (no units). The user must enter all values in a consistent system of units. The example in Figure 2 is done in lb_m, ft, sec units. All variables must be expressed in terms of these three units. This means that lb_f must not be used. Note that the spring stiffness is entered as 160·32.2. In the problem statement, the stiffness is given in lb_f/ft. However this value must be converted to lb_m/sec^2 through multiplication by g_c as shown in Figure 2(a).

A Mathcad solve block appears at the top of Figure 2(b). Within the solve block we see the differential equation itself and the two initial condition equations. Instead of ending with a "find" statement, this one ends with

$$y:=\text{Odesolve}(t,t_{max}).$$

In this statement t is, of course the independent variable, and t_{max} simply supplies the upper limit on the solution domain. The corresponding lower limit is automatically set to zero. The graph is generated by specifying t as the independent variable and y(t) as the dependent variable.

Comparison of the graph in Figure 1(b) with the graph in Figure 2(b) reveals that they are precisely the same, as they certainly should be.

Damped Free Vibration

The problem reads: *In the apparatus below, the 8.0 kg body is moved 0.20 m to the right of equilibrium and released from rest at t = 0. The viscous damping coefficient c = 20 N·s/m, and the spring constant k = 32 N/m.*

Method 1: Mathcad Applies the Initial Conditions

The differential equation that governs the motion is

$$\ddot{x} + \frac{k}{m}\cdot x + \frac{c}{m}\cdot \dot{x} = 0,$$

and the solution for under-damped motion is

$$x = C \cdot e^{-\frac{c}{2m}} sin\,(\omega_d \cdot t + \Upsilon),$$

where ω_d is the damped natural frequency.

The process of obtaining the damped solution is fundamentally different from the process of obtaining the undamped solution. Parts (a), (b), and (c) of Figure 3 illustrate the use of Mathcad to evaluate the constants in the under-damped solution. There are, of course, two other cases to be considered: critically-damped and over-damped motion. Figure 3 pertains only to the under-damped case. Separate solutions must be generated for the other two cases.

Method 2: Mathcad Solves the Differential Equation

When Method 2 is used, the transition from undamped to damped motion is easy, and the three different cases of damped motion are handled in one solution. Moving from undamped to damped motion simply involves adding the damping term to the differential equation as shown in Part (b) of Figure 4. This differential equation pertains to all three cases of motion. Depending on the assigned value of the damping coefficient c, the graph in Part (b) of Figure 4 will show under-damped, critically-damped, or over-damped vibration. This is a significant advantage of Method 2. The graph at the bottom of Figure 4(b) shows the displacement as a function of time.

Damped Forced Vibration

Method 2: Mathcad Solves the Differential Equation

A periodic forcing function is now added to the damped free vibration problem above. This is easily done by adding the forcing function term to the differential equation as shown in Part (b) of Figure 5. The graph at the bottom of Figure 5(b) shows the displacement as a function of time.

Conclusions

Mathcad's inability to deal with units under Odesolve is a significant shortcoming. While it seldom leads to incorrect answers to SI problems, great care must be exercised when using US units.

Clearly Method 2 does not exercise the student's knowledge of the process of solving differential equations. Depending on one's point of view, this may or may not be an issue in a dynamics course. Both methods do exercise the student's knowledge of dynamics. Each solution culminates with a graph of displacement versus time, which is, in the author's view, the essential point.

Method 2 does not provide an explicit equation for position as a function of time. In this method, one can easily obtain a precise value for position at any particular time. For instance, to find the displacement at t = 0.5sec, one simply enters the following statement anywhere below the solve block

$$y(0.5):=$$

and the value of the displacement will appear.

The most significant advantage of Method 2 over Method 1 is the easy transition from undamped free vibration to damped free vibration to damped forced vibration. One simply adds the appropriate terms to the differential equation that governs the motion. Furthermore, the three cases of damped vibration are handled in one solution.

A significant advantage of either computer-aided method over traditional paper and pencil methods is the ease with which various parameters (mass, spring stiffness, initial position, and initial velocity) can be varied. Observation of the effect of these parameters on the resulting motion helps the student develop insight into the vibration process. It is easily observed that changing the initial velocity affects the amplitude but not the frequency of oscillation, while changing the mass affects both.

The author finds the Method 2 approach to be preferable. The lecture time available for vibrations is very limited, and students can absorb the material much more quickly by avoiding the details of finding the algebraic solutions to the equations of motion.

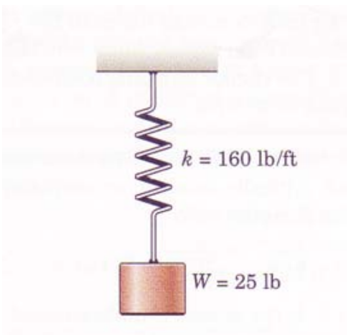
Biographical Information

Edwin G. Wiggins holds BS, MS, and Ph.D. degrees in chemical, nuclear, and mechanical engineering respectively from Purdue University. He is the Mandell and Lester Rosenblatt Professor of Marine Engineering at Webb Institute in Glen Cove, NY. Ed is a past chairman of the New York Metropolitan Section of the Society of Naval Architects and Marine Engineers (SNAME) and a past regional vice president of SNAME. As a representative of SNAME, Ed Wiggins served on the Technology Accreditation Commission, the Engineering Accreditation Commission, and the Board of Directors of the Accreditation Board for Engineering and Technology (ABET). A Centennial Medallion and a Distinguished Service Award recognize his service to SNAM.

Given: A body with mass 25 lbm is suspended from a spring with k = 160 lbf/ft. At time zero, it is 0.1 ft below the static equilibrium position with a downward velocity of 2.0 ft/s.

Find: The position of the body as a function of time

Sketch:



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Solution:

$\text{mass} := 25\text{lbm}$

$k := 160\frac{\text{lbf}}{\text{ft}}$ spring stiffness

$V_0 := 2\frac{\text{ft}}{\text{s}}$ initial velocity

$x_0 := 0.1\text{ft}$ initial position measured from static equilibrium

Initial estimates of the constants C and Ψ

$C := 1\text{ft}$ $\Psi := 1\text{rad}$ $t := 0$

Figure 1(a)
Undamped Free Vibration
Method 1 Mathcad Worksheet - Part 1

Now a Mathcad slove block to evaluate the constants

Giver

$$x_0 = C \cdot \sin\left(\sqrt{\frac{k}{mass}}\, t + \Psi\right)$$ initial condition for position

$$V_0 = \frac{d}{dt}\left(C \cdot \sin\left(\sqrt{\frac{k}{mass}}\, t + \Psi\right)\right)$$ initial condition for velocity

$$\begin{pmatrix} C \\ \Psi \end{pmatrix} := \mathrm{Find}(C, \Psi)$$

$C = 0.172 \cdot ft$ $\Psi = 0.622 \cdot rad$ actual values of constants

Therefore

$t := 0s, 0.01s .. 1s$ range variable for time

$$x(t) := C \cdot \sin\left(\sqrt{\frac{k}{mass}}\, t + \Psi\right)$$ equation for position

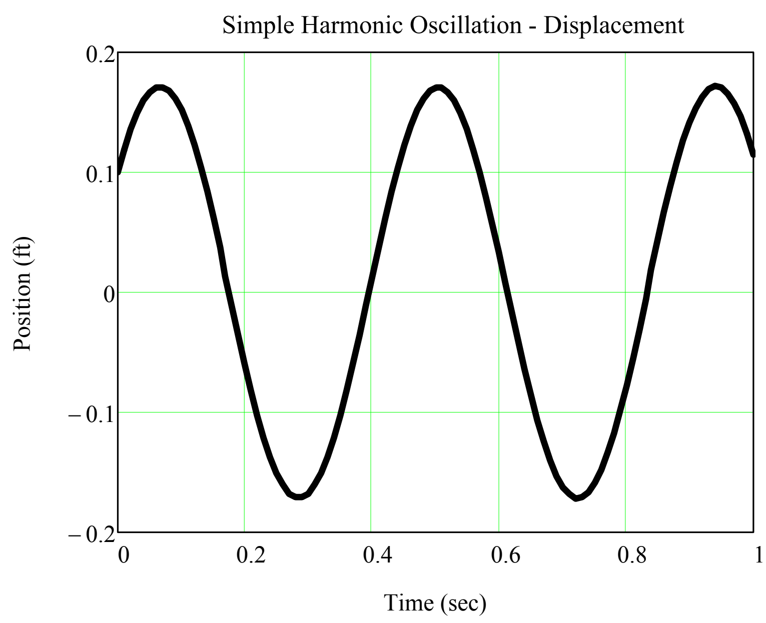
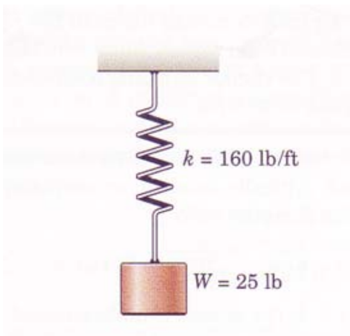


Figure 1(b)
Undamped Free Vibration
Method 1 Mathcad Worksheet - Part 2

Given: A body with mass 25 lbm is suspended from a spring with k = 160 lbf/ft. At time zero, it is in the static equilibrium position with a downward velocity of 2.0 ft/s.

Find: The position of the body as a function of time

Sketch:



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Solution:

Odesolve will not deal with units, so all parameters must be unitless. The units must be totally consistent, so lbf/ft in the spring stiffness must be converted to lbm/s^2

$t_{max} := 1$ upper limit on time in seconds

$t_0 := 0$ starting time in seconds

$k := 160 \cdot 32.2$ spring stiffness in lbm/s^2

$mass := 25$ mass in lbm

Figure 2(a)
Undamped Free Vibration
Method 2 Mathcad Worksheet - Part 1

Giver

$$y''(t) = \frac{-k}{mass} \cdot y(t)$$ the equation of motion

$$y(t_0) = 0.1$$ initial condition on position

$$y'(t_0) = 2$$ initial condition on velocity

$$y := \mathrm{Odesolve}(t, t_{max})$$

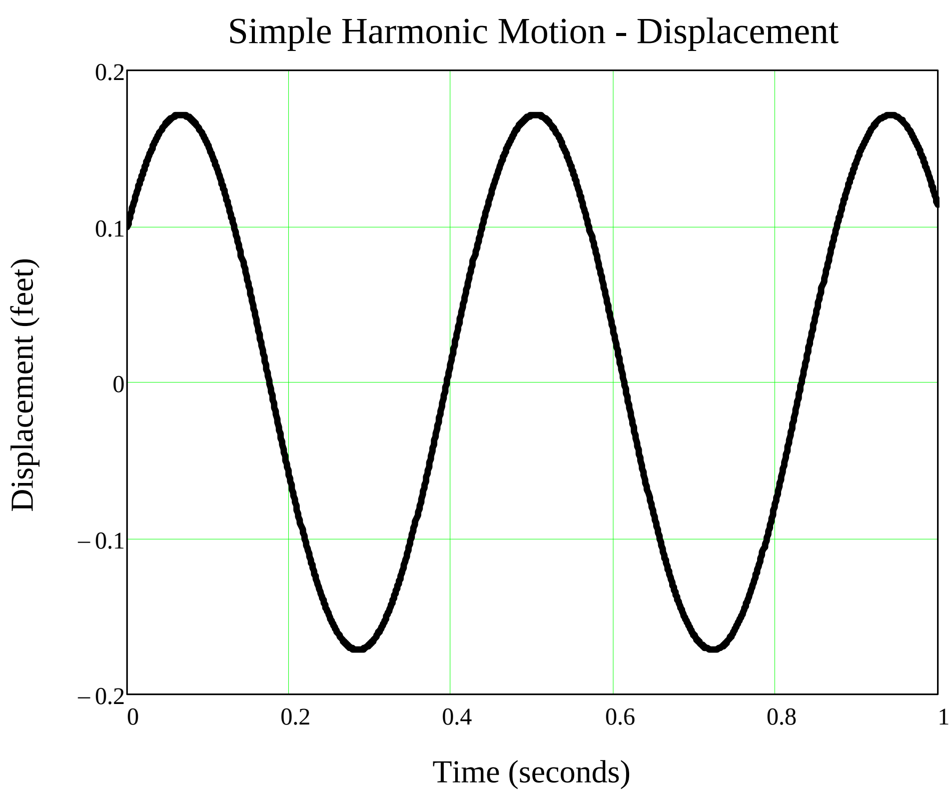
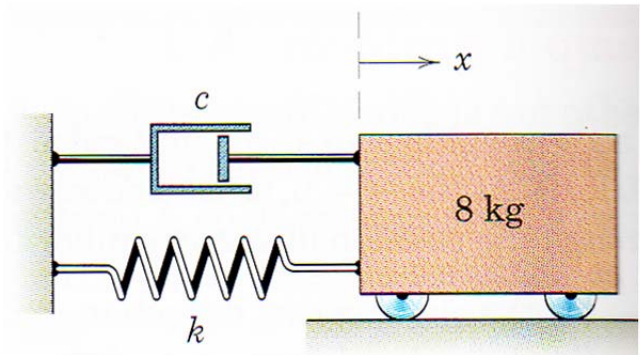


Figure 2(b)
Undamped Free Vibration
Method 2 Mathcad Worksheet - Part 2

Given: In the apparatus below, the 8.0 kg body is moved 0.20 m to the right of equilibrium and released from rest at t = 0. The viscous damping coefficient c = 20 N·s/m, and the spring constant k = 32 N/m

Find: The displacement as a function of time.

Sketch:



Solution:

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$$\text{mass} := 8\text{kg} \qquad c := 20.0\frac{\text{N}\cdot\text{s}}{\text{m}} \qquad k := 32\frac{\text{N}}{\text{m}}$$

First, determine if the system is underdamped, critically damped or overdamped.

$$\omega_n := \sqrt{\frac{k}{\text{mass}}} = 2\cdot\frac{\text{rad}}{\text{s}} \qquad \zeta := \frac{c}{2\cdot\text{mass}\cdot\omega_n} = 0.625$$

Since ζ is less than 1, we know that the system is underdamped, which means it oscillates. The damped natural frequency is therefore

$$\omega_d := \omega_n\cdot\sqrt{1-\zeta^2} = 1.561\cdot\frac{\text{rad}}{\text{s}}$$

The expression for position is therefore

$$x(t) = C\cdot e^{-\zeta\cdot\omega_n\cdot t}\cdot\sin\left(\omega_d\cdot t + \varphi\right)$$

Figure 3(a)
Damped Free Vibration
Method 1 Mathcad Worksheet - Part 1

And the velocity is the derivative of this

$$V(t) = \frac{d}{dt}x(t)$$

The initial conditions are

$$x_0 := 0.20\text{m} \qquad V_0 := 0.0\frac{\text{m}}{\text{s}}$$

We need a slove block to evaluate the constants C and φ. Of course this means we need initial estimates.

$$C := 1\text{m} \qquad \varphi := 1\text{rad} \qquad t := 0$$

Giver

$$x_0 = C \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin(\omega_d \cdot t + \varphi)$$

$$V_0 = \frac{d}{dt}\left(C \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin(\omega_d \cdot t + \varphi)\right)$$

$$\begin{pmatrix} C \\ \varphi \end{pmatrix} := \text{Find}(C, \varphi)$$

$$C = 0.256\,\text{m} \qquad \varphi = 0.896 \cdot \text{rad}$$

Therefore

$$x(t) := C \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin(\omega_d \cdot t + \varphi)$$

$$t := 0\text{s}, 0.01\text{s} .. 5\text{s}$$

Figure 3(b)
Damped Free Vibration
Method 1 Mathcad Worksheet - Part 2

Now make a graph of x versus time.

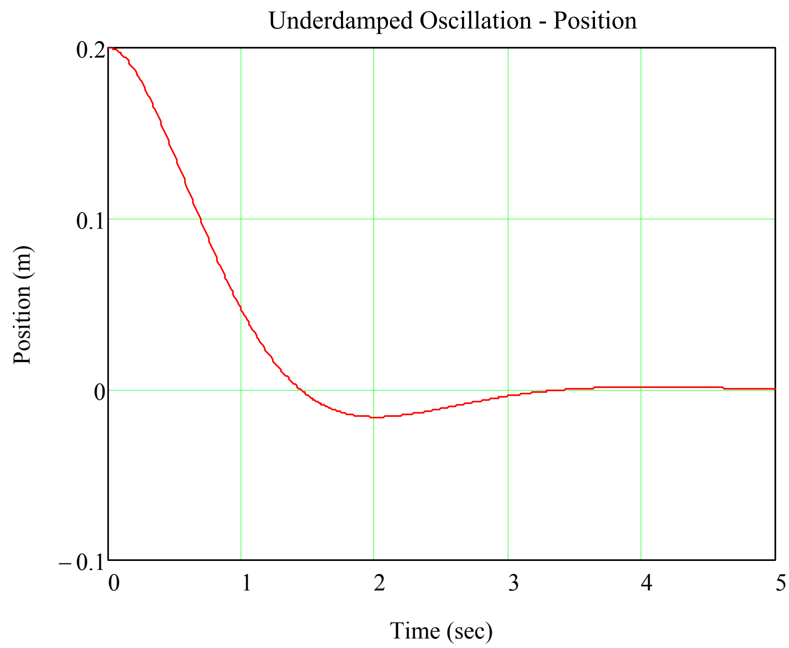
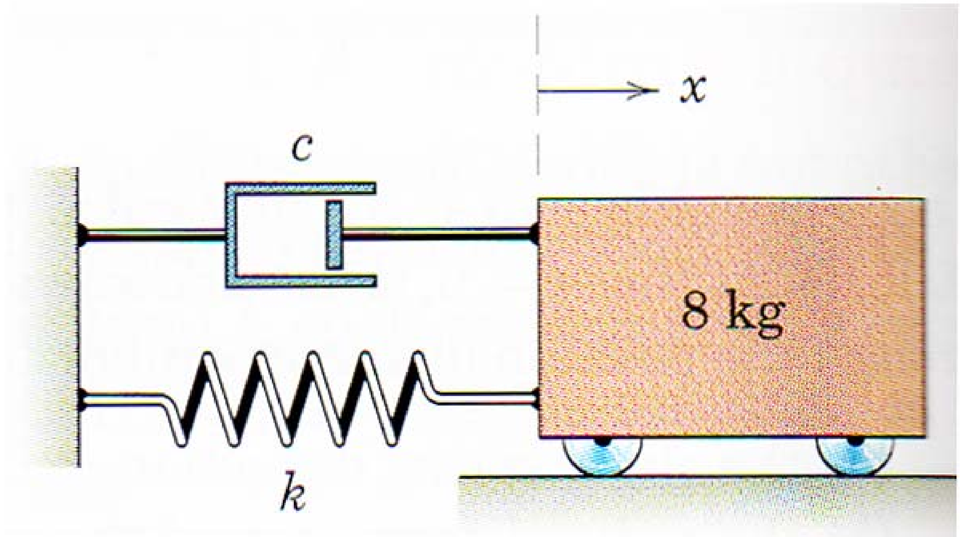


Figure 3(c)
Damped Free Vibration
Method 1 Mathcad Worksheet - Part 3

Given: In the apparatus below, the 8.0 kg body is moved 0.20 m to the right of equilibrium and released from rest at t = 0. The viscous damping coefficient c = 20 N·s/m, and the spring constant k = 32 N/m

Find: The displacement as a function of time.

Sketch:



Solution:

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Because Odesolve can not handle units, the parameters below are pure numbers in a consistent system of units (SI).

$mass := 8.0$ mass in kg

$x_0 := 0.20$ initial displacement in m

$k := 32$ spring stiffness in kg/s^2

$c := 20$ damping coefficient in kg/s

$t_{max} := 5$ upper limit on time in seconds

$t_0 := 0$ starting time in seconds

Figure 4(a)
Damped Free Vibration
Method 2 Mathcad Worksheet - Part 1

Giver

$$y''(t) = \frac{-k}{mass} \cdot y(t) - \frac{c}{mass} \cdot y'(t)$$ the equation of motion

$$y(t_0) = 0.2$$ initial condition on position

$$y'(t_0) = 0$$ initial condition on velocity

$$y := Odesolve(t, t_{max})$$

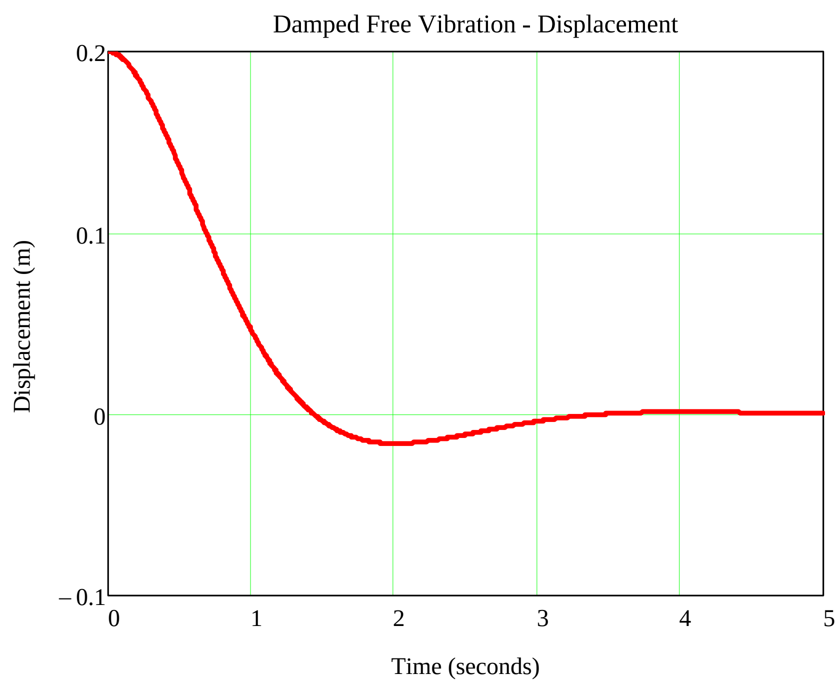
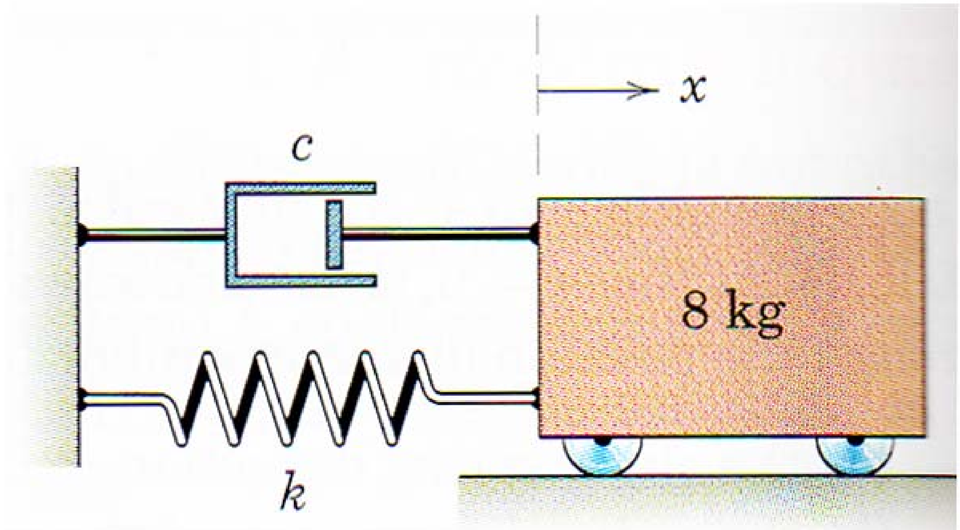


Figure 4(b)
Damped Free Vibration
Method 2 Mathcad Worksheet - Part 2

Given: In the apparatus below, the 8.0 kg body is moved 0.20 m to the right of equilibrium and released from rest at t = 0. The viscous damping coefficient c = 20 N·s/m, and the spring constant k = 32 N/m. A periodic forcing fuction of the form $F_0 \sin(\omega \cdot t)$ is applied.

Find: The displacement as a function of time.

Sketch:



Solution:

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Because Odesolve can not handle units, the parameters below are pure numbers in a consistent system of units (SI).

mass := 8.0	mass in kg
x_0 := 0.20	initial displacement in m
k := 32	spring stiffness in kg/s^2
c := 20	damping coefficient in kg/s
t_{max} := 10	upper limit on time in seconds
t_0 := 0	starting time in seconds

Figure 5(a)
Damped Forced Vibration
Method 2 Mathcad Worksheet - Part 1

$F_0 := 10$ magnitude of forcing function in kg·m/s^2

$\omega := 6$ frequency of forcing function in radian/s

Giver

$$y''(t) = \frac{-k}{mass} \cdot y(t) - \frac{c}{mass} y'(t) - \frac{F_0 \cdot \sin(\omega \cdot t)}{mass}$$ the equation of motion

$y(t_0) = 0.2$ initial condition on position

$y'(t_0) = 0$ initial condition on velocity

$y := \text{Odesolve}(t, t_{max})$

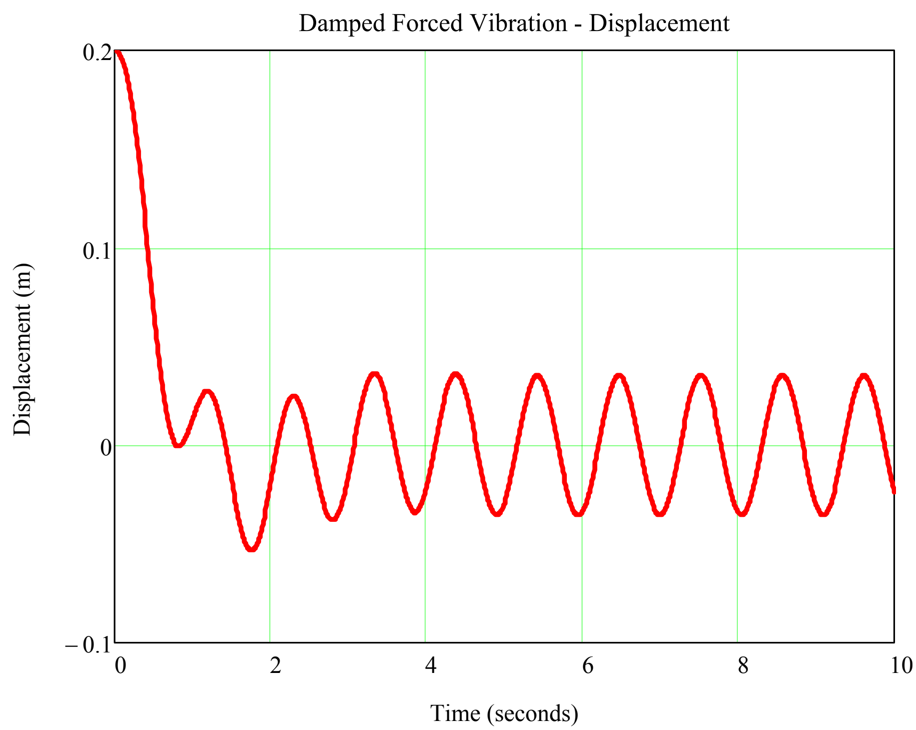

Figure 5(b)
Damped Forced Vibration
Method 2 Mathcad Worksheet - Part 2