SOLVING THE THREE PERSON GAME IN GAME THEORY USING EXCEL

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Abstract

We built templates to assist in the solving of the three person game for both total and partial conflict games. These solutions find any pure strategie solutions for the players playing alone and without communication. Then every combination of coalitions between players is found and solved. Users must interpret the results to determine if any coalition is more likely than others to be formed. Additionally, the user must consider the use of bribes or side payments to change the outcomes if any player would prefer a different outcome than the results found.

Key words: Total conflict games, partial conflict games, zero sum, nonzero sum games, game theory, Nash equilibrium, prudential strategies, security levels, Nash arbitration, linear programming, nonlinear programming, MS-Excel

Introduction

In our interdisciplinary Department of Defense Analysis at the Naval Postgraduate School, we teach a three course sequence in mathematical modeling for decision making. In the first course, we usually teach basic linear programming both using the two-variable graphical simplex technique and the Excel Solver using SimplexLP.

In this 3rd course, we teach the basic concepts and solution techniques for game theory. In our class we use the Straffin text [8] as well as Chapter 10 from Giordano, Fox, and Horton [4]. We do not cover the basic solution techniques in this paper other than to illustrate the movement diagram. In the course for total conflict games we first present the zero-sum and constant sum two-person games. We cover many of the solution techniques. Our last lesson in the block extends the two-person total conflict games to the three-person total conflict games.

In the three-person games, we find Nash equilibrium via movement diagrams and then break the game down into possible coalitions. This pits two players versus the third player. All possible coalitions are evaluated and their results are used to look for likely forming coalitions.

Next, we visit the partial conflict games. After covering the techniques for finding equilibrium and negotiated solutions, we return to the three person games. We cover the solution techniques for finding the Nash equilibrium and all the possible coalitions to attempt to determine what might happen.

Our students must complete a course project of their own choice using one of the modeling techniques from class. Students use the modeling process in their project: they identify the problem; they list the appropriate assumptions with justifications; they explain why their modeling technique is selected; they solve the model; interpret the solution; perform sensitivity analysis (if applicable); and they discuss strengths and weaknesses of their modeling approach. The three person games add more reality to many of their projects. Here is a short list of some of the game theory projects:

- Game Theory with US and Non-State actors.
- Game Theory in Cameroon-Nigeria dispute.
- Game Theory in PMI and US military tasks.
- US-China Game.
- The Somali Pirates game.
- US-Afghanistan drug dilemma.

- US-Afghanistan Regional Game.
- US Coin Operations Game.
- Dealing with Safe Havens as a Game.
- IEDS and Counter-IEDS as a Game.
- Game theory for Courses of Combat Actions
- Game Theory and Dark Money Networks
- Dealing with ISIS
- Russia, US, and Ukraine
- Dealing with Snowden as a three person game: Snowden, US Government, & Russia.

In the past, our coverage did not cover much linear programming or nonlinear programming, so our solution processes were limited to two-person, two strategy games using the algebraic method or other short cut methods because of the complexity of the solution mechanics. Recently, we have added more applications of linear programming and a non-linear template as a solution technique so students might add more reality to the number of possible strategies available to the players.

Let's define a generic simultaneous three person game theory payoff matrix as shown in Table 1. We give Larry two strategies {L1, L2}, Colin two strategies {C1, C2} and Rose two strategies {R1, R2}.

In a three person total conflict game (zero-sum or constant sum), the values in each triplet, (R_i, C_i, L_i), sum to either zero or the same constant.

In a three person nonzero-sum game the values in each triplet, (R_i, C_i, L_i), do not all sum to zero nor do they sum to the same constant.

We also make the following assumptions about the game:

Games are simultaneous
Players are rational meaning they want the best outcome possible versus their opponents.
Games are repetitive.
Players have perfect knowledge about their opponents.

3-Person Total Conflict Games

The solution methodology of the three person total conflict games involves several steps. First, we use the movement diagram, as we will describe, to find all the Nash equilibriums. The Nash equilibrium is defined when no player would unilaterally change their outcomes.

Consider the following three person (total conflict) zero-sum game between Rose, Colin, and Larry (from Straffin, Chapter 19) shown in Table 2.

	Larry L1				Larry L2		
			Colin				
		C1	C2			C1	C2
Rose	R1	(R_1,C_1,L_1)	(R_1,C_2,L_1)		R1	(R_1,C_1,L_2)	(R_1,C_2,L_2)
	R2	(R_2,C_1,L_1)	(R_2,C_2,L_1)		R2	(R_2,C_1,L_2)	(R_2,C_2,L_2)

Table 1. Generic three person game between Rose, Colin, and Larry.

		Larry L1	
		Colin	
		C1	*C2*
Rose	*R1*	*(1,1,-2)*	*(-4,3,1)*
	R2	*(2,-4,2)*	*(-5,-5,10)*

		Larry L2	
		Colin	
		C1	*C2*
Rose	*R1*	*(3,-2,-1)*	*(-6,-6,12)*
	R2	*(2,2,-4)*	*(-2,3,-1)*

Table 2. Three person game example (Source: Straffin, Chapter 19).

Movement Diagram

We define a movement diagram as follows for each player's possible outcomes R1 or R2, C1 or C2, and L1 or L2, draw an arrow from the smallest to the largest value. For Rose arrows are drawn vertically from smaller to larger. For example, under Larry L1 and Colin C1, the value 2 in R2 is greater than the value 1 in R1 so the arrow goes from R1 to R2. For Colin, arrows are drawn horizontally between C1 and C2 from smaller values to larger values. For Larry, arrows are drawn diagonally to represent the two games, L1 and L2 with arrows drawn from corresponding positions. This is illustrated in Figure 1.

		Larry L1	
		Colin	
		C1	C2
Rose	R1	(1,1,-2)	(-4,3,1)
	R2	(2,-4,2)	(-5,-5,10)

		Larry L2	
		Colin	
		C1	C2
Rose	R1	(3,-2,-1)	(-6,-6,12)
	R2	(2,2,-4)	(-2,3,-1)

Figure 1. Movement diagram for three person zero sum game.

We follow the arrows. If any set or sets of arrows bring us to a point where no arrow leaves that point or points then we have an equilibrium point or points. Result: The movement diagram reveals two pure strategy Nash equilibriums at R1C1L2 (3,-2,-1) and at R2C1L1 (2,-4, 2). These are not equivalent and not interchangeable. Going for one equilibrium point over another by either player may lead to a non-equilibrium outcome because of player's preferences.

Coalitions Possible

Let's consider communications with the ability to form coalitions. Assume first that Colin and Larry form a coalition against Rose. The following steps are helpful in the setting up and analysis of the coalition.

Step 1. Build a payoff matrix for Rose against the Colin-Larry coalition using Rose's values from the original payoffs as follows:

		Colin-Larry			
		C1L1	*C2L1*	*C1L2*	*C2L2*
Rose	*R1*	*1*	*-4*	*3*	*-6*
	R2	*2*	*-5*	*2*	*-2*

Step 2. Try to find a solution for the Nash equilibrium using either: a) Saddle points (*maximin*) or b) Mixed strategies.

a) No saddle point solution RowMin {-6, -5} ColMax {2, -4 ,3, -2}

b)

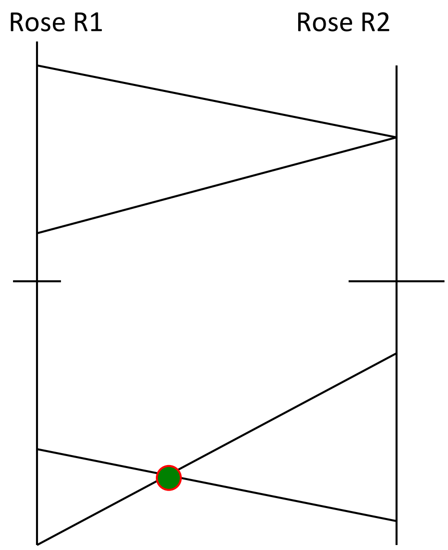


Figure 2. William's graphical method to eliminate strategies (rows) not used to obtain the solution.

The graph, Figure 2, shows that the *Maximin* solution is found by using the following values for Rose versus the Coalition. We can easily find the solution.

If the game has a saddle point solution, those values are the value of the game for all three players. Since we have a mixed strategy then we must find the value for each of our three players.

		Colin-Larry			
		C2L1	C2L2	Oddments	
Rose	R1	-4	-6	2	3/5
	R2	-5	-2	3	2/5
Oddments		1	4		
		4/5	1/5	Value	-22/5 or -4.4

Step 3. Finding the values of the game for each player.

$$\frac{3}{5}\cdot\frac{4}{5}R1C2L1+\frac{3}{5}\frac{1}{5}R1C2L2+\frac{2}{5}\frac{4}{5}R2C2L1+\frac{2}{5}\frac{1}{5}R2C2L2$$

We now substitute the values from the original payoff matrix.

$$\frac{3}{5}\cdot\frac{4}{5}(-4,3,1)+\frac{3}{5}\cdot\frac{1}{5}(-6,-6,12)+$$

$$\frac{2}{5}\cdot\frac{4}{5}(-5,-5,10)+\frac{2}{5}\cdot\frac{1}{5}(-2,3,01)$$

= (-4.4, -0.64, 5.04)

We find the payoffs are to Rose -4.4, to Colin -.64, to Larry 5.04

Step 4. Redo steps 1-3 for Colin versus a coalition of Rose-Larry and then redo steps 1-3 for Larry versus a coalition of Rose-Colin.

Results are as follows:

Colin versus Rose-Larry: Value of (2, -4, 2) and this was the saddle point solution.
Larry versus Rose-Colin (2.12, -0.69, -1.43)
Rose versus Colin-Larry (-4.4, -0.64, 5.04) from before.

Step 5. Determine which coalition, if any, yields the best payoff for each player.

Rose: Max { 2, 2.12, -4.4} is 2.12 so Rose prefers a coalition with Colin.
Colin: Max {-4,-0.69,-0.64} is -0.64 so Colin prefers a coalition with Larry.
Larry: Max {2, -1.43, 5.04) is 5.04 so Larry prefers a coalition with Colin.

In two of these cases we find that Colin-Larry is the preferred coalition so we might expect that the Colin-Larry coalition will naturally be the coalition formed. We note that we may or may not be able to determine which coalition might be formed. We also note that there are both bribes and side payments allowed. These bribes or payments entice a coalition to either change or keep the coalition together.

Characteristic function: The number *v(S)*, called the value of S, is to be interpreted as the amount S would win if they formed a coalition. We assume that the empty coalition (none are formed) value is zero, $v(\varnothing)=0$

Colin versus Rose-Larry (2,-4,2)
Larry versus Rose-Colin (2.12,-.69,-1.43)
Rose versus Colin-Larry (-4.4,-.64,5.04)

We can build the functions:

Empty set: $v(\varnothing)=0$
Alone: v(Rose) = -4.4, v(Colin) = -4, v(Larry) = -1.43
Coalition by two(s):
v(Rose-Colin) = 1.43 v(Rose-Larry) = 4
v(Colin-Larry) = 4.4

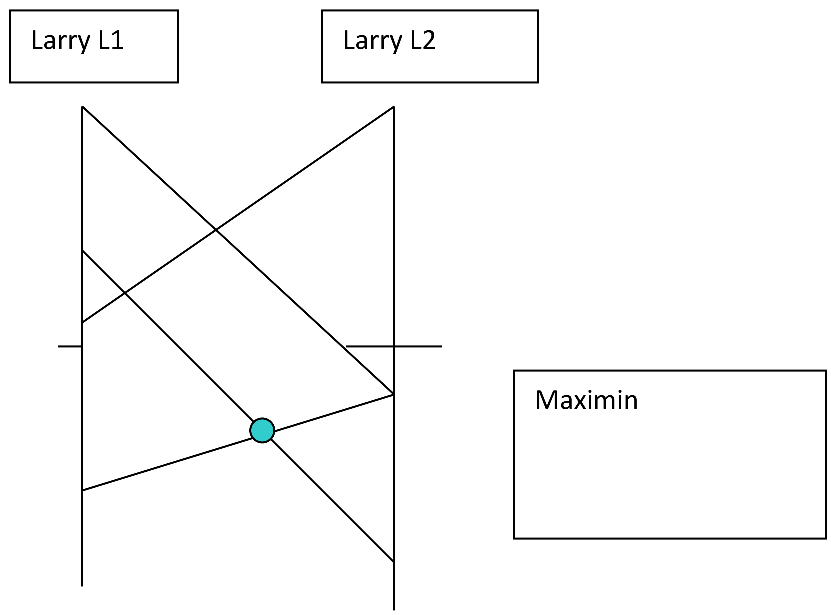
We add the payoff for the coalition's partners in the associated games.

Coalitions by three: These are zero-sum games so adding all payoffs together =0
v(Rose-Colin-Larry) = 0

Thus,
Larry versus Rose-Colin (2.12,-.69,-1.43)

		Rose-Colin			
		R1C1	*R1C2*	*R2C1*	*R2C2*
Larry	*L1*	*-2*	*1*	*2*	*10*
	L2	*-1*	*12*	*-4*	*-1*

No saddle point exists since Max of {-2,-4} is -2 and Min of {-1,12,2,10} is -1. We move on to find the mixed strategies.



Subgame:

			Rose-Colin		
		R1C1	R2C1	Oddments	
	L1	-2	2	4	3/7
Larry	L2	-1	-4	3	4/7
Oddments		1	6		
		6/7	1/7	Value is	-10/7

(3/7)*(6/7)* (1,1,-2) + (3/7)*(1/7)*(2,-4,2)+(4/7)*(6/7)*(3,-2,-1)+(4/7)*(1/7)*(2,2,-4) = (104/49, -34/49 , -10/7)

=(2.12,-.069,-1.43) rounded to 2-decimal places.

Although the mathematics is not difficult the number of calculations is quite tedious. Therefore, we built a technology assistant for student use.

Technology Assistant with EXCEL

We developed a technology assistant to assist the students with the many calculations involved. Instructions are provided within the template, which is a macro-enhanced Excel worksheet. These instructions include:

(1) Put the R,C,L entries into the blocks to the left

(2) Go to Coalition_R_CL and execute the Solver

(3) Go to Coalition_C_RL and execute the Solver

(4) Go to Coalition_L_RC and execute the Solver

(5) List the equilibrium values if the players play alone and the equilibriums in the three coalitions

(6) Determine if any coalition naturally forms

(7) Is there a legitimate bribe to change the coalition?

In Figure 3, we find the results or outcomes of the calculations made to find the pure strategies equilibrium and the results of the coalitions. The user must then interpret the results and make conclusions about those results as to what is likely to occur.

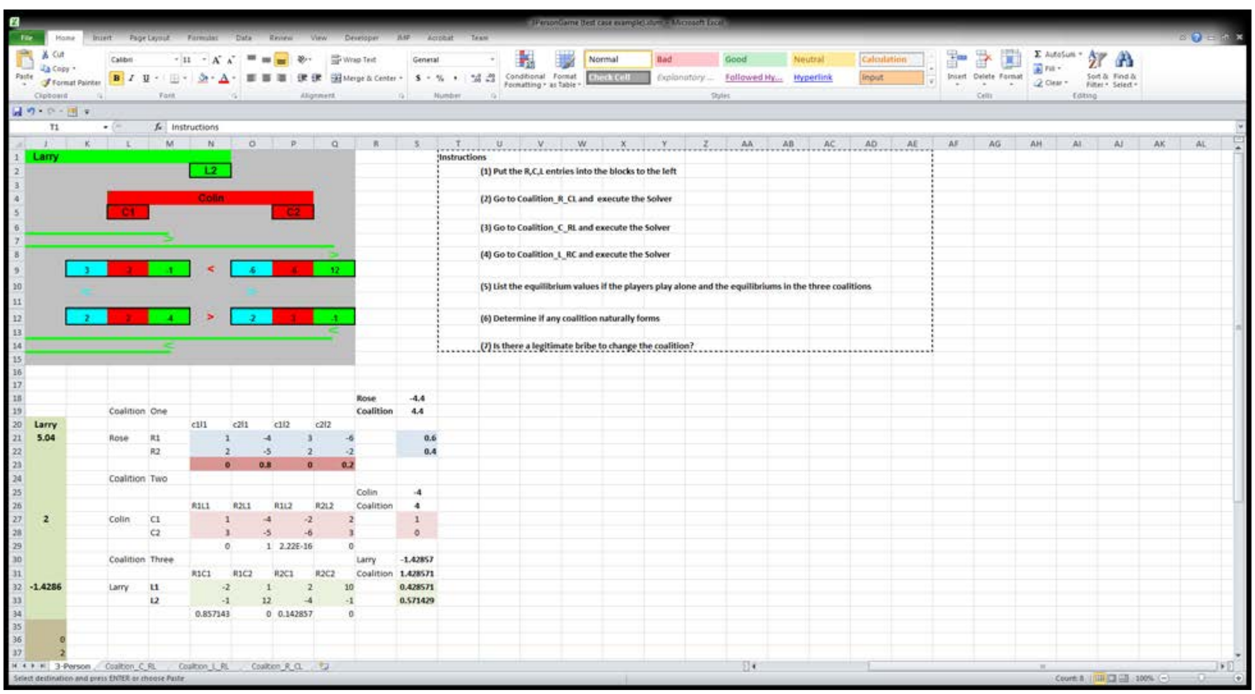


Figure 3. Screen shot of 3 person game template with instructions.

N-Person Games with Linear Programming

The coalition's solution on each worksheet uses the Solver, specifically SimplexLP. We illustrate with a three person zero-sum game that we just saw in the previous example. Recall, we created the game payoffs for the potential coalitions:

		Colin-Larry			
		C1L1	*C2L1*	*C1L2*	*C2L2*
Rose	*R1*	*1*	*-4*	*3*	*-6*
	R2	*2*	*-5*	*2*	*-2*

This coalition game is a zero sum game. We solve for Rose's solution in Excel with SimplexLP. We get the Colin-Larry coalition's results from the sensitivity column in Excel. Note there are some negative entries as payoffs so we let $v = V1\text{-}V2$ (see Winston *1995)*. We formulate the LP.

Maximize $v= V1-V2$
$x1+2x1-V1+V2\geq 0$
$-4x1-5x2-V1+V2\geq 0$
$3x1+2x2-V1+V2\geq 0$
$-6x1-2x2-V1+V2\geq 0$
$x1+x2=1$
$x1\leq 1$
$x2\leq 1$
non-negativity

We find the LP solution to this game for Rose is v =-4.4, when x_1 = 0.6 and x_2 = 0.4. We find from the reduced costs (the dual solution for Colin & Larry coalition), is V_{cl}=4,4, when $y1=y3=0$, $y2 = 0.8$ and $y4=0.2$.

Although this gives us a Coalition value, we must use all the probabilities for the players to obtain the values to each of our players separately. We only have to use the strategies with probabilities greater than 0:

(.6)(.8) R1C2L1 +(.4)(.8) R2C2L1 + (.6)(.2) R1C2L2 + (.4)(.2) R2C2L2

.48 (-4,3,1) + .32 (-5,-5,10) +.12 (-6,-6,12) + .08 (-2,3,-1) = (-4.4, -0.64,5.04)

Rose loses -4.4 (as shown before) and the Coalitions 4.4 is broken down as -0.64 for Colin and 5.04 for Larry.

We repeat this process for each Coalition to obtain these results:

Colin vs Rose-Larry (2, -4, 2)

Larry vs Rose-Colin (2.12,-0.69,-1.43)

It is still up to the user to interpret and analyze these results. These procedures work for constant sum games as well.

A 3-person Game that is a Strict Non-Zero Sum Game Using Technology

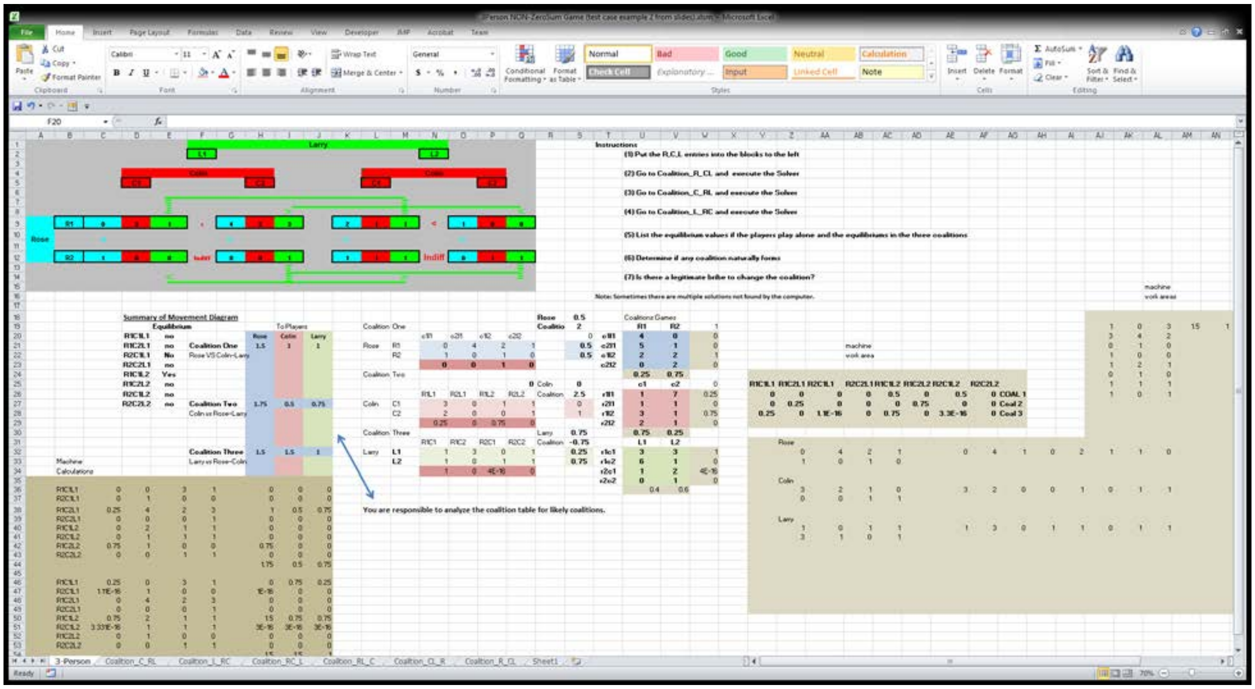
We also developed an assistant for the partial conflict game. This technology assistant requires the use of the Solver six times in the spreadsheet since each player or side in a coalition requires a linear programming solution. The instructions are listed inside the template.

The results here are as follows:

Pure strategy by movement diagram finds an equilibrium at R1C1L2 with values (2,1,1)

	Equilibrium
R1C1L1	No
R1C2L1	No
R2C1L1	No
R2C2L1	No
R1C1L2	Yes
R1C2L2	No
R2C1L2	No
R2C2L2	No

We easily see a better set of values as an output of (4,2,3) at R1C2L1. We analyze all coalitions to see if that solution rises from any coalitions.



From the linear programming solution of the coalitions, we find:

	To Players		
	Rose	Colin	Larry
Coalition One Rose VS Colin-Larry	1.5	1	1
Coalition Two Colin vs Rose-Larry	1.75	0.5	0.75
Coalition Three Larry vs Rose-Colin	1.5	1.5	1

Rose prefers a coalition with Larry, Colin prefers a coalition with Rose, and Larry prefers either a coalition with Colin or being alone. There is no preferred coalition and none gets us to the better value.

Perhaps all the players should just all agree to play the strategies that provide the best solution.

Conclusions

We have described the use of Excel templates to assist in the solution to the three person games. We remark that users must still analyze the numerical values to determine what will most likely happen. The author will provide these templates upon request. Email requests to wpfox@nps.edu.

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