THE COMPLETE FAMILY OF CONVOLUTION FORMS FOR LINEAR TIME INVARIANT SYSTEMS

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Abstract

The most common convolution technique for evaluating the output of linear time invariant systems is to convolve the system *h(t)* impulse response with the input *x(t)*. A major extension of these concepts is presented whereby the convolution of the n^{th} derivative (or integral) of the input *x(t)* can be convolved respectively with the n^{th} integral (or derivative) of the *h(t)* impulse-response to yield the output. This extension of convolution theory not only leads to more powerful techniques for system analytical analyses, it also provides the basis for an elegant mathematical interpretation of representing *x(t)* signal models as expansion of *x(t)* into an infinite sum of infinitesimal singularity functions.

Index Terms

Convolution, convolution evaluations, impulse response, singularity functions, and singu-larity response functions.

Introduction

TO formulate the complete family of convolution forms, the concept of the complete family of singularity functions $\delta_n(t)$ must be defined and presented. With this background a complete family of singularity response functions $h_n(t)$ and a generalized family of *one-sided Green's Function(s)* $g_n(t)$ are defined. Then the concept of representing an *x(t)* input as an expansion of an infinite number of infinitesimal $\delta_n(t)$ components is presented where the index *n* does not mean for all *n*, rather the expansion is for one value of *n*; e.g., *x(t)* can be expanded into $\delta_2(t)$ ramp components, or $\delta_1(t)$ step components, etc. Once this concept is understood, it becomes a simple process to use convolution as a tool to apply superposition to evaluate the total output *y(t)* which is equal to the sum of the $h_n(t)$ components where $h_n(t)$ is the response to each $\delta_n(t)$ component of the input *x(t)*. A direct result of these formulations is that an analyst now has the flexibility of solving for the output of a linear time-invariant system by convolving $h_n(t)$ with the nth derivative of *x(t)* rather than being restricted to the one common approach defined by $y(t) = h(t)*x(t)$. This flexibility, in many cases, drastically reduces the efforts required for evaluating system outputs using convolution techniques.

Finally, examples are used to illustrate the power of the extended convolution concepts and appropriate conclusions are presented with an emphasis on summarizing and clarifying the area of analytically evaluating convolution forms.

Background

The complete family of singularity functions is defined by the following relationships:

The continuous-time step-function $\delta_1(t)$ is defined by

$$\delta_1(t) = \begin{cases} 1 & t > 0 \\ 0 < \delta_1(0) < 1 & t = 0 \\ 0 & t < 0 \end{cases} \tag{1}$$

from which the complete family of the $\delta_n(t)$ singularity functions are defined as

$$\frac{d\delta_n(t)}{dt} = \delta_{n-1}(t) \quad \text{(2a)} \qquad \text{or} \qquad \delta_n(t) = \int_{-\infty}^{t} \delta_{n-1}(\tau)d\tau \quad \text{(2b)}$$

The term "or" between (2a) and (2b) reflects the fact that either one of these relationships is derivable from the other. It is also important to know that the derivative in (2a) is defined as the "derivative in the singularity sense" which means the $\delta_n(t)$ functions are represented by "distribution functions" and all derivatives exist [1]. Simons et. al. presented a graphical interpretation of the higher order derivative singularity functions such as the impulse $\delta(t)$ or $\delta_0(t)$, the doublet ($\dot{\delta}(t)$ or $\delta_{-1}(t)$), the triplet ($\ddot{\delta}(t)$ or $\delta_{-2}(t)$), … etc. [2]. The $\delta_1(t)$ step, $\delta_2(t)$ ramp, $\delta_3(t)$ parabola, etc. are easily interpreted with simple integral calculus. DeRusso et. al. in their 1965 text recognized the requirement to define and use the complete family of singularity functions in any comprehensive signals and systems text [3].

There are several good arguments to support the fact that the authors $\delta_n(t)$ definitions represent a substantial improvement over this and other early attempts to define the family of singularity functions. Not only does $\delta_n(t)$ for $n = 0$ imply $\delta_0(t) = \delta(t)$, the universal definition of the impulse, the singularity response function $h_n(t)$ and $g_n(t)$ definitions of (4) and (6) naturally follow. Alternatively, the choice of $u_0(t) = u(t)$ for the step-function would imply $u_{-n}(t) = \overset{(n-1)}{\delta}(t)$, which would be rather disconcerting. Another argument for using the $\delta_n(t)$ singularity function definition is that the n^{th} derivative of $\delta(t)$ or $\overset{(n)}{\delta}(t) = \delta_{-n}(t)$, which means that there is never a need to use negative n indices in time-domain $\delta_n(t)$ functions. Rather, use $\delta_n(t)$ for n positive, use $\delta(t)$ for $n = 0$, and use $\overset{(|n|)}{\delta}(t)$ for n negative. Finally, the Laplace-transform of $\delta_n(t)$ is simply $1/s^n$, which is another very desirable feature for choosing the $\delta_n(t)$ definitions.

With the complete $\delta_n(t)$ family of singularity functions defined, the complete family of singularity systems responses can be defined. For example, consider the *SdE* (*Standard differential Equation*) with the $v(t)$ term added

$$\sum_{n=0}^{N} b_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} a_m \frac{d^m x(t)}{dt^m} \triangleq v(t) \tag{3}$$

which is simply a single-input single output constant coefficient *dE* model. For this system model it is common practice to define the output $y(t)$ as $h(t)$ when the input $x(t)$ is the unit impulse $\delta(t)$. From this standard definition, the complete family of system responses to singularity functions can be extended and defined by

$$h_n(t) \triangleq y(t)\Big|_{x(t)=\delta_n(t)} \tag{4}$$

which implies

$$\vdots$$

$$h_2(t) = \text{ramp-response} \tag{5}$$

$$h_1(t) = \text{step-response}$$

$$h_0(t) = h(t) = \text{the standard impulse response}$$

$$h_{-1}(t) = \dot{h}(t) = \text{the doublet response}$$

$$\vdots$$

Similarly, if the whole right-hand side of the SdE is defined as $v(t)$, then the *Complete Family of Green's Functions,* or the one-sided generalized Green's function can be defined by

$$g_n(t) \triangleq y(t)\Big|_{v(t)=\delta_n(t)} \tag{6}$$

where

$$\vdots$$

$g_2(t) =$ the Green's Function ramp-response

$g_1(t) =$ the Green's Function step-response

$g_0(t) =$ the standard one-sided Green's Function

$$\vdots$$

etc.

With the definitions of (4) and (6) and the *SdE* model of (3), it is easily shown using superposition that

$$h_n(t) = \sum_{m=0}^{M} a_m \frac{d^m g_n(t)}{dt^m} \tag{7}$$

and with simple calculus

$$\frac{dh_n(t)}{dt} = h_{n-1}(t) \quad \text{or} \quad h_n(t) = \int_{-\infty}^{t} h_{n-1}(\tau)\, d\tau \tag{8}$$

Similarly

$$\frac{dg_n(t)}{dt} = g_{n-1}(t) \quad \text{or} \quad g_n(t) = \int_{-\infty}^{t} g_{n-1}(\tau)\, d\tau \tag{9}$$

This background material combined with the linear system theory and convolution concepts will provide the basis to extend convolution techniques and define the "*Complete Family of Convolution Forms.*" The well-known ***IST***

(*Impulse Sifting Theorem*) and the less familiar ***HS-SST*** (*Harden-Simons Step-Sifting Theorem*) are essential for analytically evaluating convolution forms [4].

The basic ***HS-SST*** in mathematical form is

$$\int_{-\infty}^{\infty} \delta_1(\tau-\tau_B) f(t,\tau)\delta_1(\tau_E-\tau)d\tau = \delta_1(\tau_E-\tau_B)\int_{\tau_B}^{\tau_E} f(t,\tau)d\tau \quad (10)$$

where terms of the type $\delta_1(\tau-\tau_B) f(t,\tau)\delta_1(\tau_E-\tau)$ are one of the most commonly encountered terms when analytically evaluating convolution forms, especially when functions are piece-wise continuous. Harden and Simons first published their basic ***SSTs*** (*Step-Sifting Theorems*, meaning more than one) in 1987 and followed that in 1989 with their *Generalized Step-sifting Theorems for Signals and Systems* paper where they presented a extensive range of ***SST*** forms [5]. Harden and Simons have taught these techniques to their students for over 20 years because not only do students develop a deeper understanding of the convolution process, but the typical standard 2 to 3 page textbook examples are reduced to 3 or 4 lines that can often be written in the margin of the text.

The fundamental ***SST*** concept graphically depicted in Fig. 1 is extremely simple to interpret and understand. For example, the integrand function $f(t,\tau)$ is "turned on" at the beginning τ_B by the forward running step $\delta_1(\tau-\tau_B)$ and turned off at the end τ_E by the backwards-running step $\delta_1(\tau_E - \tau)$, as expressed in (10) and illustrated graphically in Fig. 1. The $\delta_1(\tau_E - \tau_B)$ "step" serves to produce a zero value when the τ_E end occurs before the τ_B beginning begins; i.e., there is not an overlap as depicted in Fig.1. It is important to note that τ_B and τ_E are generally functions of t. Furthermore, with a little experience, analysts are able to apply ***SSTs*** without consciously memorizing their forms or applying any graphical interpretations, rather ***SST*** forms are formulated and analytically evaluated by recognizing the integrand and "$\delta_1(t)$-limits" relationships.

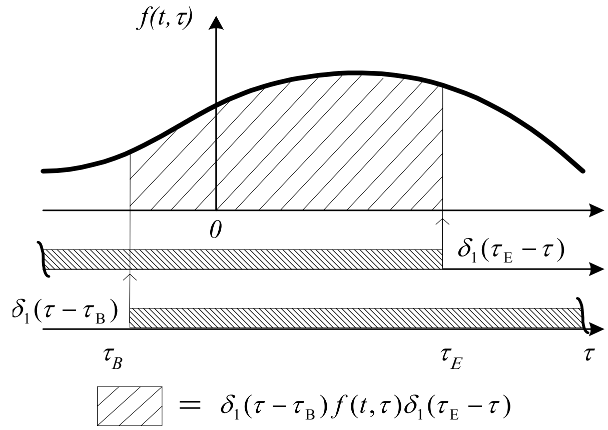

Fig. 1. A graphical representation of the *HS-SST* (Harden-Simons Step-Sifting Theorem) as expressed in equation (10).

Linearity and Convolution

If the operation L by a system on the input $x(t)$ produces an output $y(t) = L\{x(t)\}$, then the system is said to be linear iff

$$L\{a_1x_1(t)+a_2x_2(t)\} = a_1L\{x_1(t)\}+a_2L\{x_2(t)\} \quad (11a)$$

for any two inputs $x_1(t)$ and $x_2(t)$ and any scalars a_1 and a_2.

If (11a) holds for a restricted class of inputs, then the system is said to be linear for that class of inputs. The significance of (11a) is that the principle of superposition applies, i.e. evaluate the output due to each of the inputs and add them to obtain the output due to the combined inputs. Furthermore, if for a class $x(t)$ inputs

$$L\{x(t-\tau)\} = y(t-\tau) \quad (11b)$$

then the system is said to be linear time-invariant for that class of $x(t)$ inputs since the output for $x(t-\tau)$ is just a delayed version of the output for $x(t)$.

The standard convolution form for evaluating the output $y(t)$ for a linear time-invariant model is

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau \tag{12}$$

Fig. 2 can be used to explain why equation (12) is valid.

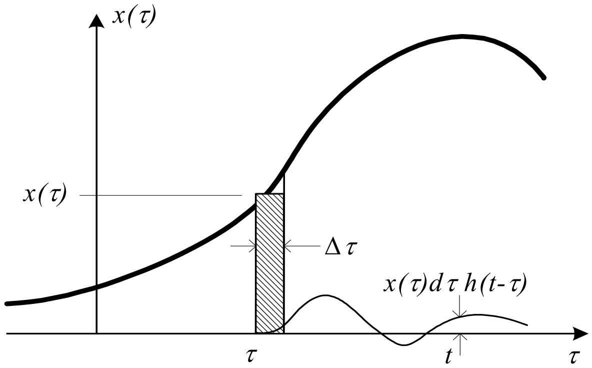

Fig. 2. The *x(t)* can be represented by a sum of $x(\tau)d\tau\,\delta(t-\tau)$ im-pulse components, each of which produces a $x(\tau)d\tau h(t-\tau)$ response.

For example, if the output due to a unit impulse is *h(t)*, then the output due to the model of the infinitesimal impulse $x(\tau)d\tau\delta(\tau)$ depicted by the shaded $x(\tau)\Delta\tau$ strip will also cause the response to be an impulse response; but this impulse response is scaled down by the area ratios $x(\tau)d\tau$ to 1 where '1" represents the unit impulse. This observation is a direct result of linearity as expressed by (11a). Thus, the magnitude of the impulse response will be proportional to the area under the impulse or the "weight" of the input impulse. If the differential response of the system output at $\tau = t$ due to an infinitesimal $x(\tau)d\tau\delta(\tau)$ impulse at any τ is defined as *dy(t)*, then

$$dy(t) = x(\tau) d\tau h(t-\tau) \tag{13}$$

is a manifestation of (11b) where the $t - \tau$ argument defines the time between the application of an infinitesimal impulse and the time the output is evaluated at $\tau = t$. Finally, the output for all of the $x(\tau)$ infinitesimal impulses becomes a matter of simple calculus or for all time *t*

$$y(t) = \int_{-\infty}^{\infty} dy(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \tag{14}$$

The familiar alternate $y(t) = h(t)*x(t)$ form for (14)

$$y(t) = \int_{-\infty}^{+\infty} x(t-\tau) h(\tau) d\tau \tag{15}$$

can be derived with a simple change of variable.

Some versions of this standard convolution derivation is common to almost all system theory texts, but what follows will focus on alternate interpretations that will lead to the *Complete Family of Convolution Forms*. To initiate the alternate interpretation, consider the common ***IST*** (*Impulse Sifting Theorem*)

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau \tag{16}$$

Based on similar arguments to what were presented for standard convolution, *x(t)* can be thought of as an expansion of *x(t)* into an infinite number of infinitesimal $x(\tau)d\tau\delta(t - \tau)$ impulses, which is also depicted in Fig. 2. The application of superposition as expressed by the definition of linearity in (11) to evaluate the summation of the outputs due to each of these $\delta(t)$ components is simply convolution. To extend this concept, consider integrating (16) by parts to obtain

$$x(t) = x(\tau)(-1)\delta_1(t-\tau)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \dot{x}(\tau)\delta_1(\tau) d\tau \tag{17}$$

and if *x(t)* is assumed to be zero as *t* approaches - ∞ , then

$$x(t) = \int_{-\infty}^{\infty} \dot{x}(\tau)\delta_1(t-\tau) d\tau \tag{18}$$

If *x(t)* is not zero as *t* approaches - ∞, then superposition can be used to treat that component separately. The form of *x(t)* in (18) can be interpreted as an expansion of *x(t)* into an infinite number of infinitesimal $\dot{x}(\tau)d\tau\delta_1(t-\tau)$ step functions as shown in Fig. 3. By induction and repeated integration by parts of (16)

$$x(t) = \int_{-\infty}^{\infty} \overset{(n)}{x}(\tau)\,\delta_n(t-\tau)\,d\tau \qquad (19)$$

which implies that *x(t)* can be represented as an infinite sum of any one $\delta_n t$ infinitesimal components; i.e., *x(t)* can be expanded into a set of $\delta_n t$ components for any one value of finite *n*. Since any input *x(t)* can be expanded into a set of infinitesimal $\delta_n t$ components, linearity implies that the output *y(t)* is equal to the sum of the outputs due to each $\delta_n t$ component. To illustrate the concept for $n = 1$, integrate (14) by parts to obtain

$$y(t) = x(\tau)(-1)h_1(t-\tau)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \dot{x}(\tau)h_1(t-\tau)\,d\tau \qquad (20)$$

which implies

$$y(t) = \int_{-\infty}^{\infty} \dot{x}(\tau)h_1(t-\tau)\,d\tau \qquad (21)$$

where *x(t)* is assumed to be nonzero in the finite range of *t*. The convolution form (21) is not new. Chen defined the inditial function, which is $h_1(t)$, and then formulated a convolution form equivalent to equation (21) [6].

To extend convolution beyond *h(t)* and $h_1(t)$, (21) can be repeatedly integrated by parts. Then the *Complete Family of Convolution Forms* for $h_n(t)$ becomes

$$y(t) = \int_{-\infty}^{\infty} \overset{(n)}{x}(\tau)\,h_n(t-\tau)\,d\tau \qquad (22)$$

In a similar manner, the following *Complete Family of Convolution Forms for Green's Functions* can be derived.

$$y(t) = \int_{-\infty}^{\infty} \overset{(n)}{v}(\tau)\,g_n(t-\tau)\,d\tau \qquad (23)$$

Equations (22) and (23) represents the *Completed Theorems of Convolution* for linear time-invariant systems. Conclusions and observations concerning the convolution forms of (19), (22), and (23) will be more meaningful after considering the following examples.

Generalized Convolution Examples

The first example is designed to provide an interpretation of how an arbitrary *x(t)* input function can be expanded into an infinitesimal set of $\delta_2 t$ ramp functions. This interpretation with the aid of Fig. 3 can be defined by

$$\begin{aligned} x(t) &= x(\tau_1) + \dot{x}(\tau_1)\delta_2(t-\tau_1) \\ &\quad + \left[\dot{x}(\tau_2) - \dot{x}(\tau_1)\right]\delta_2(t-\tau_2),\ \tau_1 < t \le \tau_2 \end{aligned} \qquad (24)$$

where effectively a ramp function $\delta_2 t - \tau_2)$ is added to *x(t)* at $t = \tau_2$ to account for change in curvature of *x(t)*. In an infinitesimal sense with τ_1 defined as the current τ

$$\lim_{\tau_2 \to \tau_1}\left[\dot{x}(\tau_2) - \dot{x}(\tau_1)\right] = d\dot{x}(\tau) = \ddot{x}(\tau)\,d\tau \qquad (25)$$

Equations (25) can be interpreted as representing $\ddot{x}(\tau)d\tau\delta_2(t-\tau)$ ramp functions, which is the special case of $n = 2$ for the general form of (19). Expansions of *x(t)* into other infinitesimal $\delta_n t$ functions can be interpreted in a similar manner.

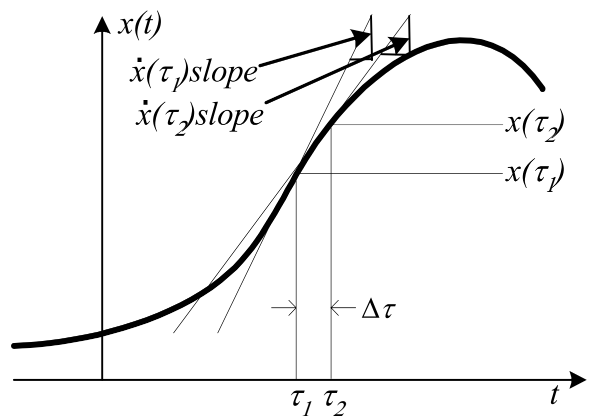

Fig. 3. An illustration of how an *x(t)* input can be represented by an infinite number of infinitesimal $\ddot{x}(\tau)d\tau\delta_2(t-\tau)$ ramp functions.

The second example is partitioned into three cases to better illustrate the concepts that have been presented. For all three cases in the second example, assume the *SdE*

$$\dot{y}(t) + 2y(t) = 3\dot{x}(t) + 4x(t) \triangleq v(t) \qquad (26)$$

to be driven by the input

$$x(t) = 10\delta_1(t) - 10\delta_1(t-5) \tag{27}$$

With standard Laplace transform techniques applied to the model defined by (26), the transform for $h_1(t)$ is

$$H_1(s) = \frac{1}{s}H(s) = \frac{1}{s}\cdot\frac{3s+4}{s+2} \tag{28}$$

which leads to

$$h(t) = h_0(t) = 3\delta(t) - 2\varepsilon^{-2t}\delta_1(t) \tag{29a}$$

$$h_1(t) = \left\{2 + \varepsilon^{-2t}\right\}\delta_1(t) \tag{29b}$$

$$h_2(t) = \left\{2t + \frac{1}{2} - \frac{1}{2}\varepsilon^{-2t}\right\}\delta_1(t) \tag{29c}$$

using Laplace techniques, equation (8), and the knowledge that $h_n(t) = 0$ for $t < 0$. The input can be expressed in the three forms

$$x(t) = 10\delta_1(t) - 10\delta_1(t-5) \tag{30a}$$

$$\dot{x}(t) = 10\delta(t) - 10\delta(t-5) \tag{30b}$$

$$\ddot{x}(t) = 10\dot{\delta}(t) - 10\dot{\delta}(t-5) \tag{30c}$$

Thus, the complete family of convolution forms

$$y(t) = h_n(t) * \overset{(n)}{x}(t) \tag{31}$$

can be applied to the system and input models of (29) and (30) for three cases: with cases 1,2, and 3 defined by n = 0, 1, and 2. The case 1 for n = 0 is defined by $h(t)*x(t)$ or

$$y(t) = \int_{-\infty}^{+\infty}\left\{10\delta_1(t-\tau) - 10\delta_1(t-5-\tau)\right\}\left\{3\delta(\tau) - 2\varepsilon^{-2\tau}\delta_1(\tau)\right\}d\tau$$
$$= 30\delta_1(t) - 30\delta_1(t-5) - 20\delta_1(t)\int_0^t \varepsilon^{-2t}d\tau + 20\delta_1(t-5)\int_0^{t-5}\varepsilon^{-2\tau}d\tau$$

where the familiar ***IST*** (*Impulse Sifting Theorem*) was used to evaluate the first two terms whereas the last two terms are determined by the ***HS-SST*** (*Harden-Simons Step-Sifting Theorem*) [4]. With simple algebra and integral calculus, the output *y(t)* can be reduced to

$$y(t) = \left\{20 + 10\varepsilon^{-2t}\right\}\delta_1(t) - \left\{20 + 10\varepsilon^{-2(t-5)}\right\}\delta_1(t-5) \tag{32}$$

The application of superposition or the given $x(t)$ implies

$$y(t) = 10h_1(t) - 10h_1(t-5) \tag{33}$$

which is shown to agree with (32) using the $h_1(t)$ of (29b). The case 2 (n = *1*) output *y(t)* is defined by

$$y(t) = \dot{x}(t) * h_1(t) \tag{34}$$
$$= \int_{-\infty}^{+\infty}\left\{10\delta(t-\tau) - 10\delta(t-5-\tau)\right\}\left\{\left[2 + \varepsilon^{-2\tau}\right]\delta_1(\tau)\right\}d\tau$$

which when evaluated using the IST results in

$$y(t) = \left\{20 + 10\varepsilon^{-2t}\right\}\delta_1(t) - \left\{20 + 10\varepsilon^{-2(t-5)}\right\}\delta_1(t-5) \tag{35}$$

Again, the previous results are consistent, and thus verified. Finally, for the case 3 where n = 2,

$$y(t) = \ddot{x}(t) * h_2(t) \tag{36}$$
$$= \int_{-\infty}^{\infty}\left\{10\left[\dot{\delta}(t-\tau) - \dot{\delta}(t-5-\tau)\right]\right\}\left\{\left[2\tau + \tfrac{1-\varepsilon^{-2t}}{d}\right]\delta_1(\tau)\right\}d\tau$$

This particular form is not common and will in most cases not be handled correctly because several literature sources claim that the product rule does not apply to impulses, doublets, etc., which is not true. The Zadeh and DeSorer text is correct in its treatment of $\delta_n(t)$ functions. The basic problem with the claim that the product rule does not apply for $\delta(t), \dot{\delta}(t)$, etc, is the failure of the analyst to recognize that the term $f(\tau)\dot{\delta}(t-\tau)$ has both an impulse and doublet component. To illustrate the point consider the term

$$w(\tau) = f(\tau)\delta(t-\tau) = f(t)\delta(t-\tau) \quad (37)$$

from which the product rule implies

$$\dot{w}(\tau) = \dot{f}(\tau)\delta(t-\tau) + f(\tau)\dot{\delta}(t-\tau) \quad (38)$$

where $\dot{f}(\tau)$ is defined as the derivative with respect to τ, not t. If the sampled version of the (37) is differentiated with respect to τ, then

$$\dot{w}(\tau) = f(t)\dot{\delta}(t-\tau) \quad (39)$$

Equating the correct versions of (38) and (39) results in

$$f(t)\dot{\delta}(t-\tau) = \dot{f}(\tau)\delta(t-\tau) + f(\tau)\dot{\delta}(t-\tau) \quad (40a)$$

which in turn reveals that

$$f(\tau)\dot{\delta}(t-\tau) = f(t)\dot{\delta}(t-\tau) - \dot{f}(\tau)\delta(t-\tau) \quad (40b)$$

or the term $f(\tau)\dot{\delta}(t-\tau)$ has a doublet and impulse term. This result can be confirmed by interpreting a simple graphical approximation of $f(\tau)\dot{\delta}(t-\tau)$. Similar to (40a)

$$\begin{aligned} f(t-5)\dot{\delta}(t-5-\tau) &= \dot{f}(t-5)\delta(t-5-\tau) \\ &\quad + f(\tau)\dot{\delta}(t-5-\tau) \end{aligned} \quad (41)$$

The relationships of (40) and (41) along with the relation-ship defined by

$$\int_{-\infty}^{\infty} \dot{\delta}(t-\tau)\delta_1(\tau)d\tau = \quad (42)$$

$$-\delta(t-\tau)\delta_1(\tau)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \delta(t-\tau)\delta_1(\tau)d\tau = \delta_1(t)$$

are needed to evaluate the more difficult $h_2(t)*\ddot{x}(t)$ form of (36). A straight forward application of (40), (41), and (42) will reveal an answer identical to other convolution forms $h(t)*x(t)$ and $h_1(t)*\dot{x}(t)$. Care must be exercised to recognize that the variable of integration is τ not t.

The third example for illustrating the power of the author's *Completed Theorems of Convolution* for linear time-invariant system analysis was presented by Goldberg et. al. [9] and repeated by Cavicchi [10] where he claimed his method was more efficient. The problem was to evaluate $y(t) = h(t)*x(t)$ where

$$h(t) = x(t) = 4\delta_2(t) - 4\delta_2(t-4) - 4\delta_1(t-4) \quad (43)$$

and the subscripts on x and h were dropped to take advantage of the authors notation. With the definitions

$$x_1(t) = 4\delta_2(t) - 4\delta_2(t-4) \text{ and } x_2(t) = 4\delta_1(t-4) \quad (44)$$

and the application of linearity

$$y(t) = h(t)*x_1(t) - h(t)*x_2(t) \quad (45)$$

The author's *Completed Theorems of Convolution* allows the reformulation of (45) into the form

$$y(t) = \int_{-\infty}^{\infty} h_2(t)\ddot{x}_1(t-\tau)d\tau + \int_{-\infty}^{\infty} h_1(t)\dot{x}_2(t-\tau)d\tau \quad (46)$$

which is reduced to the trivial case of evaluating impulse sifted integrals. The integration and differentiation of the x and h terms are also trivial in that indices are simply incremented and decremented to perform integration and differentiation on $\delta_n(t)$ functions.

Conclusions

A set of more relevant and important conclusions based on what has been presented is enumerated below.

1. Any real-world $x(t)$ signal model will possess all orders of derivatives in the singularity sense, which means that $x(t)$ can be written as

$$x(t)=\int_{-\infty}^{\infty}\overset{(n)}{x}(\tau)\delta_n(t-\tau)\,d\tau \qquad (47)$$

where *x(t)* can be interpreted as an infinite number of infinitesimal $\delta_n(t)$ components.

2. With the application of linearity, the output *y(t)* of a linear time-invariant system can be expressed as

$$y(t)=\overset{(n)}{x}(\tau)*h_n(t)=\int_{-\infty}^{\infty}\overset{(n)}{x}(\tau)h_n(t-\tau)\,d\tau \qquad (48)$$

which is the summation of the outputs due to each of the infinitesimal $\delta_n(t)$ components defined in Conclusion 1. Equation (48) is defined as the *Complete Family of Convolution Forms* (Chen's *Superposition Integrals* terminology may be an excellent alternative to *Convolution Forms*).

3. A natural extension of (48) is the *Complete Family of Convolution Forms for Green's Functions* defined by

$$y(t)=\overset{(n)}{v}(t)*g_n(t)=\int_{-\infty}^{\infty}\overset{(n)}{v}(t-\tau)g_n(\tau)\,d\tau \qquad (49)$$

where the input $v(t)$ is defined in (3). In addition, equation (7) can be to evaluate $h_n(t)$ from $g_n(t)$, which is important in that $g_n(t)$ is usually easier to solve for than $h_n(t)$.

4. The *Completed Theorems of Convolution* as expressed by (48) and (49) combined with the ***HS-SST*** (*Harden-Simons Step-Sifting Theorems*) repre-sents major extensions in convolution theory; e.g.:
First, the typical two to three page textbook examples are reduced to simple two to four line solutions. Goldberg et. al. support this claim [9].
Second, taking advantage of the generalized forms

$$y(t)=h_n(t)*\overset{(n)}{x}(t)$$

almost always simplifies the evaluations, sometimes to the point of being trivial, see examples two and three. Example three was in both Cavicchi's and Goldberg's (et. al.) papers [9,10].
Third, answers are in one composite form and defined for all *t*, which is much more desirable than having one line for each functional form and its limits. Harden and Simons first published their basic ***SSTs*** (*Step-Sifting Theorems*, meaning more than one) in 1987 and followed that in 1989 with their *Generalized Step-sifting Theorems for Signals and Systems* paper where they presented a extensive range of ***SST*** forms [4,5].

5. Cavicchi failed to recognize in his "Simplified Method" that the SST approach of Harden and Simons does not require an evaluation for the last segment of piece-wise continuous convolved functions. To illustrate consider the example 3 answer expressed in its usual form

$$y(t)=f_1(t)\delta_1(t)+f_2(t)\delta_1(t-4)+f_3(t)\delta_1(t-8)$$

The $f_3(t)$ term serves to cancel the $f_1(t)$ and $f_2(t)$ terms for $t\geq 8$ where $y(t)=0$. Thus, the answer

$$y(t)=\{f_1(t)\delta_1(t)+f_2(t)\delta_1(t-4)\}\,\delta_1(8-t)$$

is perfectly valid. The only reason for including $f_3(t)$ or any end term would be to serve as a check.

6. Cavicchi's "Simplified Method" is valuable as an alternative tool although our students find the *Complete Family of Convolution Forms* coupled with the ***HS-SST*** much easier to master. It is also noteworthy to recognize the significance of Cavicchi's work in that one of the authors, Roberts, has derived a multidimensional extension of his work, which implies further work will evolve.

7. Obviously completed convolution forms requires the usual conditions of integral existence and differenti-ability of integrand functions, a subject too broad to cover in detail in this paper. However, consider

$$L\left\{\overset{(n)}{x}(t)*h_n(t)\right\}=s^nX(s)\cdot\frac{1}{s^n}H(s)=X(s)H(s)$$

which implies the *Complete Family of Convolution Forms* applies to any *x(t),h(t)* pairs that have one-sided Laplace transforms. Similarly, it also applies to any *x(t),h(t)* pairs that have Fourier transforms. Thus, broad ranges of functions can be analyzed with these completed theorems of convolution.

Finally, the authors have developed analogous techniques and extensions for convolution in discrete system theory.

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Fred O. Simons Jr. received his BSEE degree in August, 1960 from Mississippi State University, and MS and Ph.D. degrees in electrical engineering from the University of Florida in 1962 and 1965, respectively. He joined the faculty at the University of Florida, University of Central Florida, and the FAMU-FSU College of Engineering in 1964, 1972, and 1984 respectively. He has served as administrator Directors, Head of a Navy Lab, and Departmental Chair. His primary areas of research are DSP, algorithm optimization and development, and system theory.

Richard C. Harden is a beloved colleague who published some 80+ papers with Professor Simons when they worked together from 1968 to 2001. He passed away in January 2001. He received a BME degree in 1944, a BEE in 1956, and MSE and Ph.D. degrees in electrical engineering in 1957 and 1961, respectively, all from the University of Florida. He had an illustrious career serving as a Navy officer, guided missile engineer, Chief Engineer, Deputy Chief, and Manager. He joined the faculty as a Professor of Electrical Engineering with the University of Missouri at Rolla and the University of Central Florida as Director of the UCF-South Orlando Campus in 1961 and 1967 respectively. On retirement in 1987, he was named Professor Emeritus. His honors include Teacher of the Year at UCF and UMR, National ASEE John A. Curtis Paper Award, and the establishment of an annual endowed ASEE CoED Award in his and Dr. F.O. Simons' name.

Aaron Robinson received his BSEE and MS degrees in Electrical Engineering from the Florida State University in 1994 and 1998 respectively. He was a project leader for the DSP group in the FAMU-FSU College of Engineering High Performance Computing and Simulations Research Laboratory. He earned his Ph.D. as a McKnight Fellow and joined Nortel for a year before becoming a professor of Electrical Engineering with the University of Memphis.

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