CURVE FITS FOR HEISLER CHART EIGENVALUES

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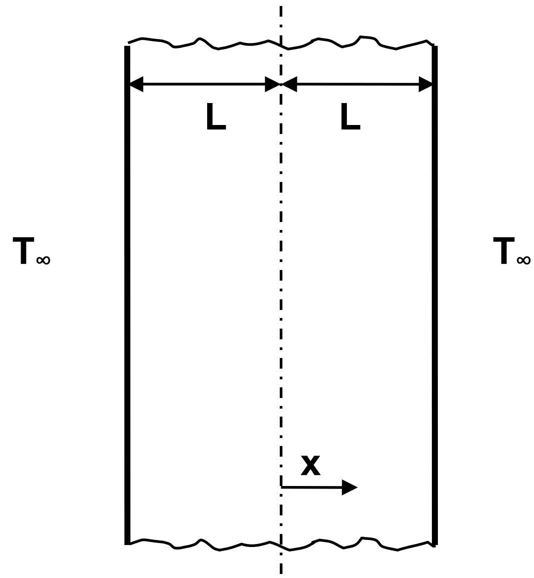
INTRODUCTION

There are many applications of transient cooling. Consider, for example, a slab of metal to be heat treated in an oven. The slab is 2L thick, and its height and width are much larger than its thickness. The slab is initially at temperature T_i (K) and is placed in an oven at T_∞ at time t = 0 (s). Both sides of the slab are heated by convection at the rate $q = 2hA_s(T_\infty - T_{x=L})$ where A_s is the surface area of one side of the slab (m^2), $T_{x=L}$ is the temperature at the surface of the slab, and h is the convection heat transfer coefficient (W/m^2-K). We need to know how long it will take for the center of the slab at x = 0 to reach a desired temperature. The temperature within the slab at a location x at time t is the solution to



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$T(x, 0) = T_i \qquad \left.\frac{\partial T}{\partial x}\right]_{x=0} = 0 \qquad -k\left.\frac{\partial T}{\partial x}\right]_{x=L} = h(T_{x=L} - T_\infty)$$

where k is the thermal conductivity of the slab material (W/m-K), ρ is the density (kg/m^3), C is its specific heat (J/kg-K), and $\alpha = k/\rho C$ is the thermal diffusivity (m^2/s).

The solution to this problem is T(x, t), but the partial differential equation (pde), boundary conditions (BCs), and initial condition (IC) have a number of parameters - k, ρ, C, h, T_i, T_∞, L . The number of parameters can be reduced by defining dimensionless variables. In this case, the following definitions work well:

$$\theta^* = \frac{T(x,t) - T_\infty}{T_i - T_\infty} \qquad x^* = \frac{x}{L} \qquad t^* = Fo = \frac{\alpha t}{L^2}$$

where Fo is the dimensionless time called Fourier number. Using these variables in the governing pde changes it to

$$\frac{\partial \theta^*}{\partial Fo} = \frac{\partial^2 \theta^*}{\partial x^{*2}}$$

and the IC and BCs become

$$\theta^*(x^*, 0) = 1 \qquad \left.\frac{\partial \theta^*}{\partial x^*}\right]_{x^*=0} = 0 \qquad \left.\frac{\partial \theta^*}{\partial x^*}\right]_{x^*=1} = -Bi\,\theta^*_{x^*=1}$$

where Bi = hL/k is the Biot number. The solution to the dimensionless problem is $\theta^*(x^*, Fo)$ subject to only 1 dimensionless parameter, Bi.

The solution to this problem is an infinite series

$$\theta^* = \sum_{i=1}^{\infty} C_i \exp(-\lambda_i^2 \, Fo) \cos(\lambda_i \, x^*)$$

where the eigenvalues λ_i are functions of Bi and are the roots of

$$\lambda_i \tan\lambda_i = Bi$$

Each C_i can be calculated once λ_i is found.

$$C_i = \frac{4\sin(\lambda_i)}{2\lambda_i + \sin(2\lambda_i)}$$

Due to the difficulty in determining the values of λ_i, the solutions to this problem have historically been presented in the form of plots called Heisler charts. In recent years, many have replaced teaching use of the Heisler charts with teaching approximate solutions that are just the first term in the exact solution series. When Fo > ≈ 0.2, which corresponds to a small value of real time, the series above reduces to only the first term.

$$\theta^* = C_1 \exp(-\lambda_1^2 Fo) \cos(\lambda_1 x^*)$$

where λ_1 and C_1 are tabulated in heat transfer texts as a function of Bi. These are tabulated because solving for λ_1 (the smallest root of $\lambda_i \tan\lambda_i = Bi$) is so difficult.

The one-term solution to this problem for a long solid cylinder and a solid sphere are very similar to the solution for a slab.

$$\theta^* = C_1 \exp(-\lambda_1^2 Fo) J_0(\lambda_1 r^*) \quad \text{cylinder}$$

$$\theta^* = C_1 \exp(-\lambda_1^2 Fo) \frac{\sin(\lambda_1 r^*)}{\lambda_1 r^*} \quad \text{sphere}$$

where $r^* = r/r_o$ and $Fo = \alpha t/r_o^2$, where r_o is the outer radius of the cylinder or sphere. λ_1 is again the first eigenvalue in a series solution and is a function of $Bi = hr_o/k$. J_0 is a zero order Bessel function of the first kind. As with a slab, the first series coefficient C_1 can be calculated once λ_1 is determined.

$$C_1 = \frac{2}{\lambda_1} \frac{J_1(\lambda_1)}{J_0^2(\lambda_1) + J_1^2(\lambda_1)} \quad \text{cylinder}$$

$$C_1 = \frac{4[\sin(\lambda_1) - \lambda_1 \cos(\lambda_1)]}{2\lambda_1 - \sin(2\lambda_1)} \quad \text{sphere}$$

where J_1 is a first order Bessel function of the first kind. The values of λ_1 and C_1 as a function of Bi are usually included in the same table with those for a slab in heat transfer texts.

Though the one-term solutions are approximations, their accuracy is better than the Heisler charts due to the inaccuracy associated with reading the charts - compounded by the fact that the charts are semilog plots. In many texts, the charts no longer appear in the main text and are placed in the appendix. The availability of Bessel functions on calculators, in spreadsheets, and in computational packages increases the utility of the one term approximate solution. However, one drawback to the approximate solutions is in determining λ_1. It must be found via the trial and error solution of a nonlinear equation for a given value of Bi, or it must be looked up in a table, often requiring interpolation. As a result, the solution to these kinds of problems is difficult to include in a program or in a spreadsheet. This paper presents easily programmable curve fits for λ_1 as a function of Bi for each of the 3 geometries - slab, cylinder, and sphere.

CURVE FITS FOR FIRST EIGENVALUE

The form used to curve fit λ_1 was not obvious to the author. A few things had been tried over the years, with unsatisfactory results. The results of a paper by Chen and Kuo (1979) suggested the appropriate form. They developed approximate solutions for θ* for all three geometries using an integral heat balance approach. The solutions they presented suggested a curve fit of the form

$$\frac{1}{\lambda_1^2} = a_0 + \frac{a_1}{Bi} + a_2 \exp\left(\frac{-a_3}{Bi}\right)$$

may work well, where a_0, a_1, a_2, and a_3 are curve fit parameters for each of the 3 geometries. Tabulated values of λ_1 (Incropera and DeWitt, 2002) were fit to this form using a nonlinear least squares method, giving the values shown in Table 1.

	a_0	a_1	a_2	a_3
slab	0.355172	0.995309	0.050421	3.550265
cylinder	0.136699	0.497324	0.036019	3.963059
sphere	0.073707	0.331704	0.027464	4.483979

Table 1. Curve Fit Parameters for Slab, Cylinder, and Sphere.

RESULTING SOLUTIONS

A comparison between the curve fit results for λ_1 and the tabulated values shows agreement to 3 or 4 decimal places with the tabulated values that are given rounded to 4 decimal places. The corresponding values of C_1 also agree to 3 or 4 decimal places compared to the tabulated values, which are also given rounded to 4 decimal places. Comparison of solutions for θ^* using tabulated values and calculated values of λ_1 and C_1 for wide ranges of Fo and Bi are shown in Tables 2a, 2b, and 2c. The values of θ^* being compared range from 1 at Fo = 0 to 0 as Fo → ∞. Differences of magnitude less than 1E-10 are shown as zero.

x*=	1.0	Fo			
		0.5	5	50	500
Bi	0.01	8.245E-05	2.981E-04	1.589E-03	1.742E-04
	0.1	8.838E-05	6.648E-04	8.612E-05	0
	1	-2.360E-04	-4.338E-05	0	0
	10	-2.170E-04	-5.698E-08	0	0
	100	-2.437E-04	-5.590E-09	0	0

Table 2a. $\theta^*_{table} - \theta^*_{curve\ fit}$ for a Slab at x* = 1.

r*=	1.0	Fo			
		0.5	5	50	500
Bi	0.01	7.676E-05	5.374E-04	2.119E-03	2.613E-06
	0.1	3.505E-04	1.133E-03	1.672E-06	0
	1	-3.742E-04	-2.299E-06	0	0
	10	-9.015E-05	0	0	0
	100	4.062E-05	0	0	0

Table 2b. $\theta^*_{table} - \theta^*_{curve\ fit}$ for a Cylinder at r* = 1.

r*= 1.0		Fo			
		0.5	5	50	500
Bi	0.01	8.747E-05	6.622E-04	1.690E-03	2.309E-08
	0.1	4.011E-04	9.473E-04	1.642E-08	0
	1	-3.902E-04	-4.882E-08	0	0
	10	-2.823E-05	0	0	0
	100	7.894E-06	0	0	0

Table 2c. $\theta^*_{table} - \theta^*_{curve\ fit}$ for a Sphere at r* = 1.

CONCLUSIONS

The curve fit form chosen gave an excellent fit to the tabulated values of λ_1 for all three geometries. The resulting approximate solutions are essentially identical to those calculated using the tabulated values, which are more accurate than reading numbers from a Heisler chart. The curve fits allow cooling calculations to be automated by making possible the creation of spreadsheet functions and program functions that eliminate the need for using charts or looking up tabulated values.

REFERENCES

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2. **Incropera, Frank P. and DeWitt, David P., Fundamentals of Heat and Mass Transfer, 5th Ed., Wiley, 2002, p 258.**

BIOGRAPHICAL INFORMATION

Dr. Rick Couvillion received his BSME at the University of Arkansas in 1975 and his PhD in mechanical engineering from Georgia Tech in 1981. He a professional engineer and started his career as a mechanical engineering faculty member at the University of Arkansas in Spring 1982. He teaches graduate and undergraduate courses in thermal sciences, computer methods, and electronic packaging. His primary research area is energy systems. Rick is a member of Tau Beta Pi and ASEE. He is a member and serves as faculty advisor for Pi Tau Sigma, ASME, and ASHRAE student chapters. He currently serves as ASME Region X assistant VP for education and is the editor of several ASHRAE Fundamentals Handbook chapters.