A COMPUTER FACILITY LOCATION: AN EXAMPLE OF OPTIMIZATION, MODELING AND SOLUTIONS WITH TECHNOLOGY

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ABSTRACT

Optimization problems are typically taught graphically in college algebra and later in calculus through differentiation. We have a straightforward optimization problem, optimal location of a facility whose set up through mathematical modeling is easy but whose solution via traditional calculus is nearly impossible. This is the type problem where technology is required to achieve solutions.

INTRODUCTION

Consider a small company that is planning to install a central computer with cable links to five departments. According to their floor plan, the peripheral computers for the five departments will be situated as shown by the dark circles in Figure 1. The company wishes to locate the central computer so that the minimal amount of cable will be used to link to the five peripheral computers. Assuming that cable may be strung over the ceiling panels in a straight line from a point above any peripheral to a point above the central computer, the distance formula may be used to determine the length of cable needed to connect any peripheral to the central computer. Ignore all lengths of cable from the computer itself to a point above the ceiling panel immediately over that computer. That is, work only with lengths of cable strung over the ceiling panels.

The coordinates of the locations of the five peripheral computers are listed in Table 1.
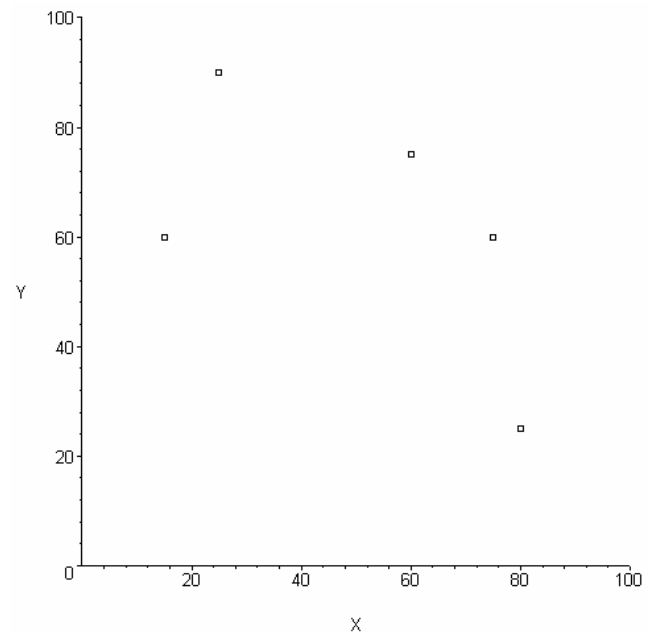

Figure 1. The Grid for the Five Departments

X	Y
15	60
25	90
60	75
75	60
80	25

Table 1. Grid Coordinates of Five Departments

Assume the central computer will be positioned at coordinates (m, n) where m and n are *integers* in the grid representing the office space. Determine the coordinates (m,n) for placement of the central computer that minimize the total amount of cable needed. Report the total number of feet of cable needed for this placement along with the coordinates (m,n).

THE MODEL

This is a multivariable optimization problem. We want to minimize the sum of the distances from each department to the placement of the central computer system. The distances represent the cable lengths assuming that the straight line is the shortest distance between two points. Using the distance formula,

$$d = \sqrt{(x - X_1)^2 + (y - Y_1)^2}$$

where d represents the distance (cable length in feet) between the location of the central computer (x,y) and the location of the first peripheral computer (X_1, Y_1). Since we have five departments we define

$$d_i = \sqrt{(x - X_i)^2 + (y - Y_i)^2}, \text{ for i} = 1,2,3,4\ 5$$

We wish to minimize the sum of the distances given by

$$Dist = \sum_{i=1}^{5} d_i = \sum_{i=1}^{5} \sqrt{(x - X_i)^2 + (y - Y_i)^2}.$$

Traditional calculus suggests taking the derivative of the function, *Dist(x,y)* first with respect to x and then with respect to y, setting the partial derivatives equal to zero, and solving for *x* and *y* to find the stationary points. Using the locations of the five peripheral computers **X=[15 25 60 75 80]** and **Y=[60 90 75 60 25]**, the objective function

Dist(x,y) becomes $Dist = \sum_{i=1}^{5} d_i$, which is

expressed below with the help of the computer algebra system, MAPLE.

Dist(x,y)=

$$\sqrt{x^2 - 30x + 3825 + y^2 - 120y} + \sqrt{x^2 - 50x + 8725 + y^2 - 180y}$$
$$+ \sqrt{x^2 - 120x + 9225 + y^2 - 150y} + \sqrt{x^2 - 160x + 7025 + y^2 - 50y}$$
$$+ \sqrt{y^2 - 120y + 9225 + x^2 - 150x}$$

The partial derivatives of the function *Dist(x,y)* with respect to *x* and *y* are:

> d4f1:=diff(d4,x);
$$\frac{\partial Dist(x,y)}{\partial x} =$$

$$dfx := \frac{2x - 30}{2\sqrt{x^2 - 30x + 3825 + y^2 - 120y}} + \frac{2x - 50}{2\sqrt{x^2 - 50x + 8725 + y^2 - 180y}}$$
$$+ \frac{2x - 120}{2\sqrt{x^2 - 120x + 9225 + y^2 - 150y}} + \frac{2x - 160}{2\sqrt{x^2 - 160x + 7025 + y^2 - 50y}}$$
$$+ \frac{2x - 150}{2\sqrt{y^2 - 120y + 9225 + x^2 - 150x}}$$

> df4f2:=diff(d4,y);
$$\frac{\partial Dist(x,y)}{\partial y} =$$

$$\frac{2y - 120}{2\sqrt{x^2 - 30x + 3825 + y^2 - 120y}} + \frac{2y - 180}{2\sqrt{x^2 - 50x + 8725 + y^2 - 180y}}$$
$$+ \frac{2y - 150}{2\sqrt{x^2 - 120x + 9225 + y^2 - 150y}} + \frac{2y - 50}{2\sqrt{x^2 - 160x + 7025 + y^2 - 50y}}$$
$$+ \frac{2y - 120}{2\sqrt{y^2 - 120y + 9225 + x^2 - 150x}}$$

These are now very complicated expressions with square roots in the denominators that are non-trivial. Closed form analytical solutions to these partial derivatives, $\frac{\partial Dist(x,y)}{\partial x} = 0$ and $\frac{\partial Dist(x,y)}{\partial y} = 0$, will not be obtained by students using traditional paper and pencil methods.

SOLUTIONS METHODS AVAILABLE WITH TECHNOLOGY

In previous work, Fox and Richardson[1] suggested other methods when traditional calculus fails to meet our needs. These options as well other heuristic approaches to obtaining "good" results will be illustrated. We note that the definition of "good" may depend upon the course in which this material is presented.

USING MAPLE AND THE SECOND PARTIAL TEST

If the stipulation requiring *integer* coordinates as solutions is relaxed, the MAPLE commands and the second partials tests can be used to find the stationary point that yields the absolute minimum in this case. The MAPLE *fsolve* command is used to solve the equations, $\frac{\partial Dist(x, y)}{\partial x} = 0$ and $\frac{\partial Dist(x, y)}{\partial y} = 0$, resulting in the stationary point (56.81841102, 68.07515715) for which $Dist(x,y) =$ 157.6634722 feet. The determinant of the Hessian matrix, the matrix of second partials, for the objective function $Dist(x,y)$ was first determined using the Maple Commands *hessian* and *det* and then evaluating at the stationary point (56.81841102, 68.07515715) with the following results:

```
> fsolve({d4f1=0,df4f2=0},{x,y});
          { x = 56.81841102, y = 68.07515715 }
>evalf(subs({x = 56.81841102,
          y = 68.07515715},d4));
               157.6634723

> with(linalg):
Warning, the protected names norm and trace
have been redefined and unprotected
>
Hessian:=subs((x=56.81841102,y=68.0751575,
matrix(2,2,[ddfxx,ddfxy,ddfxy,ddfyy])));
```

$$Hessian := \begin{bmatrix} 0.1416584422 & -0.01488222755 \\ -0.01488222755 & 0.1096294944 \end{bmatrix}$$

```
> det(Hessian);
  0.01530846270
```

Since the principal minors are all positive (0.1416584422, 0.1096294944, and 0.01530846270) then we are guaranteed a minimum number of feet of cable (157.6634722) if the central computer is located at the point represented by (56.81841102, 68.07515715) on the grid shown in figure 1. A plot of the function $Dist(x,y)$ is shown in figure 2.
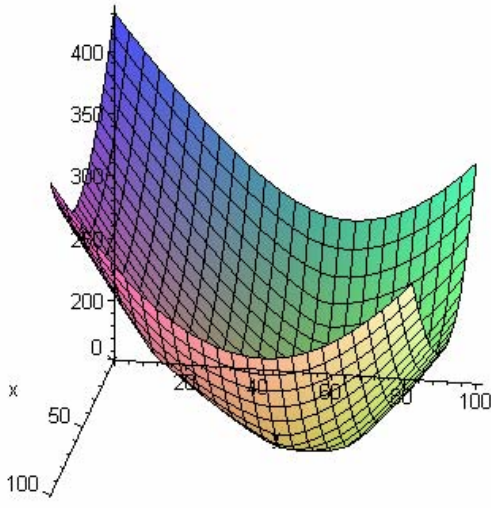

Figure 2. Plot of the Distance Function, $Dist(x,y)$

THE STEEPEST DESCENT/ASCENT ALGORITHMS

We implement a procedure in MAPLE that used the well-known method of steepest ascent to maximize a function of two variables. Since we want to minimize our function, we will multiply our function by –1 and maximize it.

Algorithm: To find a maximum solution to given a multivariable unconstrained function, f(x)

INPUT:	starting point x_0; tolerance, t
OUTPUT:	Approximate x^*, and $f(x^*)$
Step 1.	Initialize the tolerance, t >0.
Step 2.	Set $x=x_0$ and define the gradient at that point. $\nabla f(x_0)$
Step 3.	Calculate the maximum of the new function $f(x_i+t_i \nabla f(x_i))$, where $t_i \geq 0$, by finding the value of t_i.

Step 4.   Find the new x_i point by substituting t_i into
$$x_{i+1} = x_i + t_i \nabla f(x_i)$$

Step 5.   If the length (magnitude) of x, defined by
$$\| x \| = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}, \text{ is less}$$
than the tolerance specified, then continue.
Otherwise, go back to Step 3.

Step 6.   Use x* as the approximate stationary point and compute, f(x*), the estimated maximum of the function.

STOP

Because the MAPLE procedure Steepest was written as a maximization routine, the negative of the objective function must be used in order to find its maximum value, which is the true function's minimum value. The MAPLE code that follows stores the coordinates of the grid locations of the peripheral computers in vectors X and Y, defines the objective function (-1) * Dist(x,y) as function of variables x1 and x2 (needed for routine Steepest), and calls routine Steepest with the following arguments:

100= maximum number of allowed iterations
0.05= tolerance value, *tol*
(0,0) = starting point (x,y), X_0
Dist = objective function definition

Maple Input
```
>    f:=-(sqrt((x1-15)^2+(x2-60)^2)+sqrt((x1-25)^2+(x2-90)^2)+sqrt((x1-60)^2+(x2-75)^2)+sqrt((x1-80)^2+(x2-25)^2)+sqrt((x2-60)^2+(x1-75)^2));
```

$$f := -\sqrt{x1^2 - 30\,x1 + 3825 + x2^2 - 120\,x2} - \sqrt{x1^2 - 50\,x1 + 8725 + x2^2 - 180\,x2}$$
$$- \sqrt{x1^2 - 120\,x1 + 9225 + x2^2 - 150\,x2} - \sqrt{x1^2 - 160\,x1 + 7025 + x2^2 - 50\,x2}$$
$$- \sqrt{x2^2 - 120\,x2 + 9225 + x1^2 - 150\,x1}$$

```
>
(kt,MP,z1,z2,z3):=STEEPEST(100,.05,0,0,f):
```

Initial Condition: (0.0000, 0.0000)

Iter	Gradient Vector G	magnitude G	x[k]	Step Length
1	(2.8702, 3.6375)	4.6335	(0.0000, 0.0000)	19.0907
2	(.2868, -.2263)	.3654	(54.7946, 69.4424)	6.7062
3	(.0119, .0150)	.0192	(56.7180, 67.9247)	

Approximate Solution: (56.7180, 67.9247)
Maximum Functional Value: -157.6652
Number gradient evaluations: 4
Number function evaluations: 3

The values *kt, MP, z1, z2, z3* returned from routine *Steepest* are not needed here. In three iterations routine Steepest determined an approximate stationary point that is very close to the actual stationary point determined using the Second Partials Test. Here the minimum value was found to be approximately 157.6652 at the point (56.7180, 67.9247).

NEWTON'S METHOD

A second MAPLE procedure was written that uses   Newton's   method   to   optimize (maximize/minimize)   a   function   of   two variables. The specific algorithm shown here uses Cramer's rule to implement Newton's method.[2]

INPUT: x(0), y(0), N, Tolerance
OUTPUT: x(n), y(n)
Step 1. For n= 1 to N do
    Step 2. Calculate the new estimate for x(n) and y(n) as follows:

$$\frac{\partial F}{\partial x}(x(n-1), y(n-1)) \rightarrow q$$

$$\frac{\partial F}{\partial y}(x(n-1), y(n-1)) \rightarrow r$$

$$\frac{\partial G}{\partial x}(x(n-1), y(n-1)) \rightarrow s$$

$$\frac{\partial G}{\partial x}(x(n-1), y(n-1)) \rightarrow t$$

$$- F(x(n-1), y(n-1)) \rightarrow u$$

$$- G(x(n-1), y(n-1)) \rightarrow v$$

$$qt - rs \rightarrow D$$

$$x(n-1) + (ut - vr)/D \rightarrow x(n)$$

$$y(n-1) + (qv - su)/D \rightarrow y(n)$$

Step 3. If $((x(n)-x(n-1))^2 + (y(n)-y(n-1))^2)^{1/2}$ tolerance,

Then Stop
Else , Go back to Step 2.
STOP

After providing MAPLE with the vectors X and Y that contain the grid locations of the peripheral computers and defining the objective function *Dist(x,y)*, the MAPLE procedure Newtons was called with the following arguments:

100= maximum number of allowed iterations
0.05= tolerance value, *tol*
(40,50) = starting point (x,y), X_0
Dist = objective function definition

MAPLE Input:

> X:=vector([15,25,60,75,80]):
> Y:=vector([60,90,75,60,25]):
>Dist:=sum(sqrt((x1-X[i])^2+(x2Y[i])^2),i=1..5);

$$Dist := \sqrt{x1^2 - 30\,x1 + 3825 + x2^2 - 120\,x2} + \sqrt{x1^2 - 50\,x1 + 8725 + x2^2 - 180\,x2}$$
$$+ \sqrt{x1^2 - 120\,x1 + 9225 + x2^2 - 150\,x2} + \sqrt{x1^2 - 150\,x1 + 9225 + x2^2 - 120\,x2}$$
$$+ \sqrt{x1^2 - 160\,x1 + 7025 + x2^2 - 50\,x2}$$

➢ (kt,MP,z1,z2,z3):=Newtons(Dist,100,0.05,40,50):

MAPLE Output:

    initial values x= 40.000   y= 50.000

iteration 1: new x=59.1490788   new y=69.2508722
        Hessian: [.053   .008]
        eigenvalues: [.051 .089]   [.008   .088]

iteration 2: new x=57.2869316   new y=68.3324568
        Hessian: [.207   .013]
        eigenvalues: [.086   .208]   [.013 .088]
iteration 3 : new x=56.8385891   new y=68.0778746
        Hessian: [.153   -.013]
         eigenvalues: [.103 .157]   [-.013   .106]
iteration 4 : new x=56.81844271   new y=68.0751727
        Hessian: [.142 -.015]
         eigenvalues: [.104   .148] [-.015   .109]

final new x= 56.8184427   final new y= 68.0751727
final fvalue is 157.663472200

In four iterations the MAPLE procedure *Newtons* found the approximate stationary point (56.8184427, 68.07512727) corresponding to

the functional value approximately 157.6634722. The consistently positive eigenvalues provide the sufficient conditions to ensure the value (157.6634722) is a minimum for the objection function.

PROGRAMMING

Recall that initially integer coordinates that placed the central computer optimally within the grid (figure 1) were requested. Until now this fact has been ignored. A method that would address this*, integer programming*, is only

covered at the end of our mathematical programming course and would be too advanced for most undergraduate classrooms. However, integer solutions can easily be found using a short computer program that computes and compares the amount of cable needed for each of the 101 x 101 grid positions with integer coordinates (this is implicit numeration of integer programming). In fact, this heuristic technique has been used by the authors in an introductory Fortran 90 programming course for which calculus is not even a prerequisite. An illustration using MAPLE appears below.

```
> restart;
> Computer:=proc(m::integer,n::integer, f)
>   local disthold, xhold,yhold,i,j,dist;
>   disthold:=infinity;
>   for i from 0 to m do
>     for j from 0 to n do
>       dist:=evalf(f(i,j));
```

```
>     if (dist < disthold) then
>         disthold:=dist;
>         xhold:=i;
>         yhold:=j;
>       end if;
>     end do;
>  end do;
> print( xhold, yhold, disthold);
> end:
>
> X:=vector([15,25,60,75,80]):
> Y:=vector([60,90,75,60,25]):
>Dist:=(x,y)->sum(sqrt((x- X[i])^2+(y
        Y[i])^2),i=1..5);
```

$$Dist := (x, y) \rightarrow \sum_{i=1}^{5} \sqrt{(x - X_i)^2 + (y - Y_i)^2}$$

```
>Computer(100,100,Dist);
```

MAPLE Output:
 57, 68, 157.6663224

Our computer program provides the integer solution as the coordinates of *(57, 68)* with a functional value *f(57,68)= 157.6663224*.

SUMMARY

This problem has a non-integer solution of *x=56.81841102,   y=68.07515715, d=157.6663224*, which can be found using our numerical methods or by using MAPLE commands to implement the seconds partials test.   The solution *x=57*, *y=68*, and *dist= 157.663224* was determined if the integer coordinates are required. The problem, which was relatively simple to set up, quickly became difficult to solve without resorting to technology.

REFERENCES

1. Fox, William P. and William H. Richardson.. "Multivariable Variable Optimization When Calculus Fails: Gradient Search Methods in Nonlinear Optimization Using MAPLE", Computers in Education Journal, Vol. XII(4), pages 2-11, October-December 2002.

2. Meerschaert, Mark. Mathematical Modeling. Academic Press, San Diego, CA 1993, p.73.

BIOGRAPHICAL INFORMATION

Dr. William P. Fox received his BS degree from the U. S. Military Academy, his MS at the Naval Postgraduate School, and his Ph.D. at Clemson University.   He has taught mathematics at the U.S. Military Academy and at Francis Marion University. He is currently professor and chair of mathematics at Francis Marion University.   He has authored several textbooks on mathematical modeling. He has over one hundred technical and educational articles,   presentations   and   workshops presentations. He serves as the associate contest director for COMAP's Collegiate Mathematical Contest in Modeling (MCM) and serves as contest director for the High School Mathematical Contest in Modeling (HiMCM). His interests include applied mathematics, optimization, mathematical modeling, statistical models for medical research, and computer simulations.

Dr. William H. Richardson received his BS degree from Francis Marion College, his MS from Clemson University, and his Ph.D. from the Medical University of South Carolina. He is currently assistand professor at Francis Marion University.   His interests include computer programming, optimization, modeling, and photography.