

# SOLUTION OF SIMILARITY TRANSFORM EQUATIONS FOR BOUNDARY LAYERS USING SPREADSHEETS

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## ABSTRACT

A spreadsheet based solution of the similarity transformation equations of laminar boundary layer equations is presented. In this approach the nonlinear third order differential equations, for both the hydrodynamic and the thermal boundary layer equations, are discretized using a simple finite difference approach which is suitable for programming spreadsheet cells. This approach was implemented to solve the similarity transform equations for a flat plate (Blasius equations). The thermal boundary layer result was used to obtain the heat transfer correlation for laminar flow over a flat plate in the form of  $Nu = Nu(Pr, Re)$ . The relative difference between results of the present approach and those of published data are less than 1%. This approach can be easily covered in the undergraduate Fluid Mechanics and Heat Transfer courses. Also, it can be incorporated in graduate Viscous Fluid Mechanics and Convection Heat Transfer courses. Application of the present approach is not limited to the flat plate boundary layer analysis. It can be used for the solution of a number of similarity transformation equations, including wedge flow problem and natural convection problems that are covered in graduate level courses.

## INTRODUCTION

A commonly used approach for solving laminar boundary layer equations is the similarity transformation. This method converts the partial differential equations for hydrodynamic and thermal boundary layers into two third order, nonlinear ordinary differential equations, with the hydrodynamic equation for a

flat plate boundary layer known as the Blasius equation. A similarity solution of these boundary layer equations is an integral part of undergraduate Fluid Mechanics and Heat Transfer courses, as well as graduate Convection Heat Transfer, and Viscous Fluid Mechanics.

Exact solutions to the Blasius equation and other boundary layer transformation equations are quite tedious; hence only final results of the Blasius equation, in tabular form, are presented in Fluid Mechanics and Heat Transfer textbooks.<sup>1-3</sup> Numerical approaches, such as Runge-Kutta integration are commonly used to solve these equations via an iterative shooting method, since one of the boundary conditions for the differential equations is given at infinity.

It is highly desirable that students go over the entire process and obtain final numerical results, since this process makes them appreciate the solution methodology. However, due to the nature of Fluid Mechanics or Convection Heat Transfer courses most instructors do not want to cover a detailed numerical method, and students enrolled in these courses have diverse programming language skills. Some of them have a very good knowledge of computer programming, and some with very little or no programming skills. All Mechanical Engineering Students however, are familiar with spreadsheets.

This paper presents a simple approach based on spreadsheets for solving the similarity transformation equations of laminar boundary layers. The results based on this approach are identical to the exact solutions to the third

decimal place. This approach can be demonstrated in a class when covering the Blasius equation or can be assigned to the students as homework. The results associated with a thermal boundary layer for different Prandtl numbers are also correlated to obtain a correlation for Nusselt number for laminar flow over a flat plate.

### BOUNDARY LAYER EQUATIONS FOR LAMINAR FLOW

The governing equations for a steady state incompressible laminar boundary layer flow with constant fluid properties are given by<sup>1</sup> :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (3)$$

where equations (1), (2) and (3) are the continuity, momentum and energy equations, respectively. The boundary conditions for the above equations are given by:

$$\begin{aligned} u &= 0 & \text{at} & \quad y = 0, \\ u &= U_\infty & \text{as} & \quad y \rightarrow \infty \\ u &= U_\infty & \text{at} & \quad x = 0 \end{aligned} \quad (4)$$

$$\begin{aligned} T &= T_w & \text{at} & \quad y = 0, \\ T &= T_\infty & \text{as} & \quad y \rightarrow \infty \\ T &= T_\infty & \text{at} & \quad x = 0 \end{aligned} \quad (5)$$

The pressure gradient in equation (2) can be evaluated based on the fact that the pressure is constant in the y-direction, hence it can be written in terms of pressure variation in the free stream using the Bernoulli's equation,

$$\frac{\partial p}{\partial x} = -U_\infty \frac{dU_\infty}{dx} \quad (6)$$

Note that the free stream velocity for a flat plate is constant, which results in  $\partial p / \partial x = 0$ .

The velocity components can be defined in terms of the stream functions  $\psi(x, y)$ ,

$$\begin{aligned} u &= \frac{\partial \psi(x, y)}{\partial y} & \text{and} \\ v &= -\frac{\partial \psi(x, y)}{\partial x} \end{aligned} \quad (7)$$

The introduction of the stream function results in the elimination of the continuity equation (1). Considering the flow over a flat plate, application of the of the similarity transformation leads to two new dependent and independent variables,  $f$  and  $\eta$ , respectively. These variables are given by (see<sup>1-3</sup>):

$$f(\eta) = \frac{\psi(x, y)}{U_\infty \sqrt{\nu x / U_\infty}} \quad (8)$$

$$\eta = y \sqrt{U_\infty / \nu x}$$

Substituting the stream function  $\psi(x, y) = f(\eta) \sqrt{\nu x U_\infty}$  into equations

(7) and then equation (2) yields

$$2f''' + ff'' = 0 \quad (9)$$

Hence the momentum equation for the laminar boundary layer (equation (2)) is reduced to a nonlinear, third-order ordinary differential equation. Using the hydrodynamic boundary conditions given by equation (4) the boundary conditions for equation (9) become,

$$\begin{aligned} f &= f' = 0 & \text{at} & \quad \eta = 0 & \text{and} & \quad f'' = 1 \\ \text{as} & & & & & \quad \eta \rightarrow \infty \end{aligned} \quad (10)$$

Solution of equation (9) gives the boundary layer velocity profile

$$\frac{u}{U_\infty} = f' \quad (11)$$

and the wall shear stress

$$\tau_w = \mu U_\infty \sqrt{U_\infty / \nu x} \left. \frac{d^2 f}{d\eta^2} \right|_{\eta=0} = \mu U_\infty \sqrt{U_\infty / \nu x} f''(0) \quad (12)$$

The analytical solution of the Blasius equation is quite tedious. A series expansion method is presented in<sup>4</sup> and numerical integration approach is presented in.<sup>5</sup> Numerical solution of the Blasius equation is reported by a number of authors using numerical integration techniques such as the Runga-Kutta method. In addition, the numerical solutions require a good programming skill, and also since  $f''$  at  $\eta = 0$  is not known a shooting algorithm must be implemented.

In the present approach the first and second derivatives are written in a forward difference format

$$f' = \frac{f_{i+1} - f_i}{\Delta\eta} \quad (13)$$

$$f'' = \frac{f'_{i+1} - f'_i}{\Delta\eta} \quad (14)$$

$$f''' = \frac{f''_{i+1} - f''_i}{\Delta\eta} \quad (15)$$

The values of  $f, f', f''$  at  $i+1$ th step can be evaluated in terms of their values at the  $i$ th step using the equation (13) – (15) and they are given by:

$$f_{i+1} = f_i + (\Delta\eta)f'_i \quad (16)$$

$$f'_{i+1} = f'_i + (\Delta\eta)f''_i \quad (17)$$

$$f''_{i+1} = f''_i + (\Delta\eta)f'''_i \quad (18)$$

$f'''_i$  in equation (18) can be evaluated by rewriting equation (9) in the following form:

$$f'''_i = -\frac{f_i f''_i}{2} \quad (19)$$

Equations (16)- (19) are used to solve the Blasius equation using Microsoft's Excel spreadsheet program.

First, a table with headings of  $\eta, f, f', f''$  and  $f'''$  is generated in an Excel file. The first row of this table corresponds to the values of these functions at  $\eta = 0$ . Based on the boundary conditions given by equation (10)  $f = f' = 0$  at  $\eta = 0$ . The value of  $f''$  is not known at  $\eta = 0$ . Therefore, an arbitrary number is entered for  $f''(0)$ . Since  $f = 0$  at  $\eta = 0$  based on equation (19),  $f'''(0) = 0$ . Figure 1 shows the heading with initial values of the function. Note that the value of  $f''$  at  $\eta = 0$  is arbitrary set to 0.8, one may choose to enter a different number in this cell. Later the actual value of  $f''(0)$  will be computed.

	A	B	C	D	E	F	G
1							
2		<b><math>\eta</math></b>	<b><math>f</math></b>	<b><math>f'</math></b>	<b><math>f''</math></b>	<b><math>f'''</math></b>	
3		0	0.0000	0.0000	0.8000	0.0000	
4							
5							
6							
7							
8							
9							

Figure 1: Spreadsheet headings for  $\eta, f, f', f''$  and  $f'''$  with their initial values

The next step is to select a  $\Delta\eta$ . Since the order of accuracy of numerical differentiation based on equation (13) – (15) is in the order of  $\Delta\eta$  a small value of  $\Delta\eta = 0.01$  is chosen, its value is entered in cell H3 of the spreadsheet (see Figure 2). Then the following equations are entered in row 4 of the spreadsheet:

For cell B4:  $=B3+\$H\$1$   
 (this statement increases the value of cell B3 by  $\Delta\eta$ )

For cell C4:  $=C3+\$H\$1*D3$   
 (this statement is based on equation (16))

For cell D4:  $=D3+\$H\$1*E3$   
 (this statement is based on equation (17))

For cell E4:  $=E3+\$H\$1*F3$   
 (this statement is based on equation (18))

For cell F4:  $=-C4*E4/2$   
 (this statement is based on equation (19))

The resulting Excel file would then look like what is shown in Figure 2.

	A	B	C	D	E	F	G	H
1							$\Delta\eta=$	0.01
2		$\eta$	$f$	$f'$	$f''$	$f'''$		
3		0	0.0000	0.0000	0.8000	0.0000		
4		0.01	0	0.008	0.8	0		
5								
6								
7								

Figure 2: Finite difference equations are typed in row 4, and  $\Delta\eta$  is set to 0.1

The equations of row 4 are then copied and pasted into rows 5 through 803. Row 803 gives the values of  $f, f', f''$  and  $f'''$  at  $\eta=8$  when  $f''(0)=0.8$ . Noted that  $\eta=8$  is large enough to be considered as infinity. Therefore,  $f''(0)$  should be changed to force  $f'(8)=1$  in order to satisfy boundary condition  $f'(\infty)=1$  (see the boundary conditions given by equation (10)). This is accomplished by the shooting method in the numerical integration method, which a trial and error approach and its implementation can be time consuming. Excel's solver can do this easily using a few simple steps.

Excel's Solver can be accessed through "Tools" in the Menu Bar. Figure 3 shows Excel's Solver dialog box. For this problem the "Set Target Cell" in the solver parameter box should be set to cell  $\$D\$804$ , i.e. the  $f'(8)$  value. Then under "Equal To" select "Value of" and enter 1 in the corresponding textbox. For "By Changing Cells:" enter cell  $\$E\$3$ . This would force cell  $\$D\$803$  to one (i.e.  $f'(8)=1$ ) by changing value of cell  $\$E\$3$  ( $f''(0)$ ). After clicking on the "Solve" button the value of cell  $\$D\$803$  changes to one (boundary condition  $f'(\infty)=1$ ), by changing cell  $\$E\$3$  ( $f''(0)$ ), which was originally set to 0.8. After a few seconds the results converge and the dialog box that is shown in Figure 4 appears on the screen. Make sure that "Keep Solver Solution" is checked and then press "OK". The resulting table then is the solution of the Blasius equation.

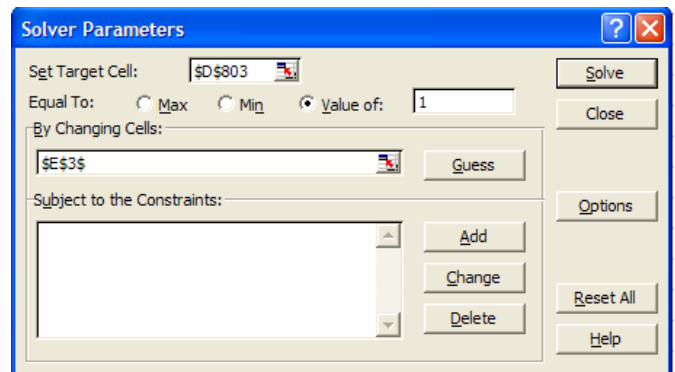


Figure 3: Excel solver dialog box for setting boundary conditions

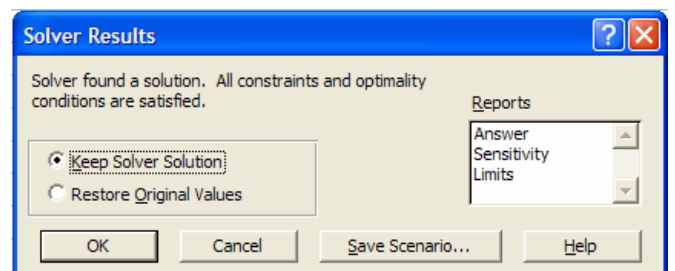


Figure 4: Solver Results dialog box

$\eta$	$f$	$f'$	$f''$	$f'''$
0	0.0000	0.0000	0.3303	0.0000
0.01	0	0.0033	0.33031	0
0.02	3.3E-05	0.00661	0.33031	-5E-06
0.03	9.9E-05	0.00991	0.33031	-2E-05
0.04	0.0002	0.01321	0.33031	-3E-05
0.05	0.00033	0.01652	0.33031	-5E-05
0.06	0.0005	0.01982	0.33031	-8E-05

Figure 5: Converged results of solver for  $\eta = 0$

An item of interest, in the table of results, is  $f''(0)$  which can be used in equation (12) to evaluate the wall shear stress. As shown in Figure 5 the calculated  $f''(0)$  based on the Excel's solver is 0.33031, while the published results based on rigorous integration methods is 0.33206 [4] (0.5% relative error). If  $\Delta\eta$  is set to 0.001 then  $f''(0)$  becomes 0.33188 which has a relative error of only 0.018% (almost the exact solution). Most textbooks report this number in three decimal places; therefore such a small difference is not noticeable. Based on the calculated  $f''(0)$  equation (12) becomes,

$$\tau_w = 0.332\mu U_\infty \sqrt{U_\infty / \nu x} \quad (20)$$

In most Heat Transfer and Fluid Mechanics textbooks the above equation is given in terms of the local friction coefficient and Reynolds number,

$$C_{f,x} = \frac{\tau_{w,x}}{\frac{1}{2}\rho U_\infty^2} = \frac{0.662}{Re_x^{1/2}} \quad (21)$$

Another important result is the velocity distribution,  $u/U_\infty = f'$  and  $f''$  which are shown in Figure 6. As can be seen in this figure the velocity is zero at the wall and  $u/U_\infty = 1$  as  $\eta \rightarrow \infty$ . Figure 6 also shows  $f''$ , which is indicative of shear stress at different layers.

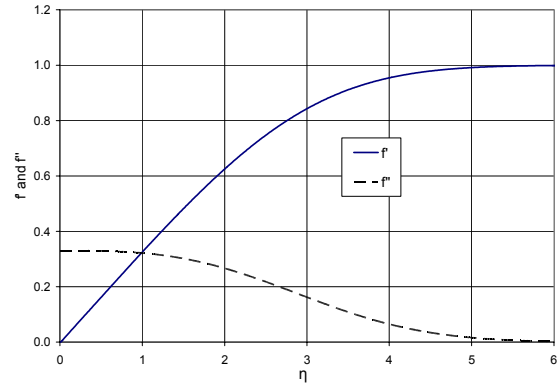


Figure 6: Velocity profile  $u/U_\infty = f'$  and  $f''$  based on the present Blasius solution

Next consideration is given to the energy equation. Application of the similarity transform to equation (3) leads to the following ordinary differential equations for the dimensionless temperature distribution,  $\theta = (T - T_w)/(T_\infty - T_w)$  as a function of  $\eta$ , i.e.  $\theta = \theta(\eta)$ , see<sup>2</sup>]for details,

$$2\theta'' + Pr f\theta' = 0 \quad (22)$$

Using the thermal boundary conditions given by equation (5) provides the boundary conditions for the dimensionless temperature distribution  $\theta$ , and are given by:

$$\theta = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \theta = 1 \quad \text{as} \quad \eta \rightarrow \infty \quad (23)$$

Note that when the Prandtl number (Pr) in equation (22) is set to unity, then the solution of the energy equation becomes the same as the Blasius equation by setting  $\theta = f'$ . The solution of the energy equation results in the local heat transfer coefficient, which is given by:

$$h_x = k \left( \frac{U_\infty}{\nu x} \right)^{1/2} \theta'(0) \quad (24)$$

The dimensionless form of the above equation is given by:

$$Nu_x = \frac{h_x x}{k} = Re_x^{1/2} \theta'(0) \quad (25)$$

To determine  $\theta'(0)$  equation (22) must be solved for different Prandtl numbers. Similar to the Blasius equation the finite difference form of the dimensionless temperatures can be written as:

$$\theta_{i+1} = \theta_i + (\Delta\eta)\theta'_i \quad (26)$$

$$\theta'_{i+1} = \theta'_i + (\Delta\eta)\theta''_i \quad (27)$$

and  $\theta''$  in equation (27) can be calculated using equation (22), i.e.,

$$\theta''_i = -\frac{Pr f_i \theta'_i}{2} \quad (28)$$

Figure 7 shows the headings and values associated with the dimensionless temperature  $\theta$  and its derivatives along with the  $f$  values. It also shows the values of  $\theta$  and its derivatives at  $\eta = 0$ , which is  $\theta = 0$ , and  $\theta'$  is arbitrary is set to 0.8. Initially, all calculations are performed for  $Pr=0.6$ , the same calculations can be repeated for other Prandtl numbers. As shown in Figure 7 the value of the Prandtl number is defined in cell  $J\$1$ . Equations (26)-(28) are entered in the cells of row 4 in the following format:

	A	B	C	D	E	F	G	H	I	J
1							$\Delta\eta=$	0.01		Pr=
2		$\eta$	$f$	$f'$	$f''$	$f'''$	$\theta$	$\theta'$	$\theta''$	0.6
3		0	0.0000	0.0000	0.33031	0.0000	0.0000	0.8000	0	
4		0.01	0	0.0033	0.33031	0	0.008	0.8	0	
5		0.02	3.3E-05	0.00661	0.33031	-5E-06				
6		0.03	9.9E-05	0.00991	0.33031	-2E-05				
7		0.04	0.0002	0.01321	0.33031	-3E-05				
8		0.05	0.00033	0.01652	0.33031	-5E-05				

Figure 7: Initial conditions and first row setup for the energy equation

For cell G4:  $=G3+\$H\$1*H3$

(this statement is based on equation (26))

For cell H4:  $=H3+\$H\$1*I3$

(this statement is based on equation (27))

For cell I4:  $=\$J\$1*C4*H4/2\$J\$$

(this statement is based on equation (28))

Cells G4, H4 and I4 are then copied and pasted into the corresponding cells from row 5 through row 803. Note that as in the hydrodynamic solution,  $\eta = 8$  corresponds to infinity. The value of cell G803 which is  $\theta(8)$  must become unity by changing  $\theta'(0)$ , see the thermal boundary conditions (equation (23)). This can be done by using Excel's solver as was done for the hydrodynamics solution. The resulting  $\theta'(0)$  then becomes 0.2758 (see Figure 9 for final results when  $Pr=0.6$ ). The value of  $\theta'(0)$  can be used in equation (25) to obtain the Nusselt number. Since  $\theta'(0)$  is dependent on the value of the Prandtl number, the same procedure repeated for different Prandtl numbers to obtain the results of  $\theta'(0)$  for various Prandtl numbers. The resulting values of  $\theta'(0)$  versus Prandtl number are shown in Table 1, and are shown in Figure 10. This graph follows a power function (i.e.,  $y = ax^b$ ). Excel can then be used to identify the trendline equation. This can be done by clicking on the "Chart" from the "Menu Bar" and then select "Add Trendline." Make sure to choose the power function and under "Options" tab check "Display equation on chart" and "Display R-squared value on chart." By clicking on the "OK" button the trendline and its equation are displayed in a power function format, as is shown in Figure 10. The  $R^2$  value of 0.9999 indicates an excellent fit to the data. The resulting curve fit equation indicates that the correlation between  $\theta'(0)$  and the Prandtl number is in the following format:

$$\theta'(0) = 0.3313 Pr^{0.3357} \quad (29)$$

Table 1: Values of  $\theta'(0)$  versus Prandtl Number

Pr	$\theta'(0)$
0.6	0.2758
0.8	0.3054
1	0.3303
1.2	0.3520
1.5	0.3802
2	0.4196
2.5	0.4527
3	0.4815
4	0.5306
5	0.5718
8	0.6691
12	0.7657
20	0.9069
50	1.2268
100	1.5400

Substituting for  $\theta'(0)$  into equation (25) yields the correlation for the Nusselt number as a function of Reynolds and Prandtl numbers, i.e.,

$$Nu_x = \frac{h_x x}{k} = 0.3313 Re_x^{1/2} Pr^{0.3357} \quad (30)$$

This correlation in the Heat Transfer textbooks is given as:

$$Nu_x = \frac{h_x x}{k} = 0.332 Re_x^{1/2} Pr^{1/3} \quad (31)$$

The results show that the difference between the correlation coefficients and Prandtl number's exponents are less than 1%.

Another important thermal boundary layer characteristics which can be obtained from the results of this approach is the temperature distribution. The resulting dimensionless temperature distributions  $\theta = (T - T_w)/(T_\infty - T_w)$  for different Prandtl

numbers are shown in Figure 11. As shown in this Figure all temperature profiles satisfy the boundary conditions given by equation (23).

G803		=G802+\$H\$1*\$H\$02							
	A	B	C	D	E	F	G	H	I
796		7.93	6.19814	1	1.4E-05	-4E-05	2.900579	0.00198	-0.00368
797		7.94	6.20814	1	1.4E-05	-4E-05	2.900599	0.00194	-0.00362
798		7.95	6.21814	1	1.3E-05	-4E-05	2.900618	0.00191	-0.00356
799		7.96	6.22814	1	1.3E-05	-4E-05	2.900637	0.00187	-0.0035
800		7.97	6.23814	1	1.3E-05	-4E-05	2.900656	0.00184	-0.00344
801		7.98	6.24814	1	1.2E-05	-4E-05	2.900674	0.0018	-0.00338
802		7.99	6.25814	1	1.2E-05	-4E-05	2.900692	0.00177	-0.00332
803		8	6.26814	1	1.2E-05	-4E-05	2.90071	0.00174	-0.00326
804									

Figure 8: Last rows values shown  $\theta(8)$  in cell G802

	$\Delta\eta=0.01$					Pr=0.6		
$\eta$	f	f'	f''	f'''	$\theta$	$\theta'$	$\theta''$	
0	0.0000	0.0000	0.33031	0.0000	0.0000	0.2758	0	
0.01	0	0.0033	0.33031	0	0.002758	0.27579	0	
0.02	3.3E-05	0.00661	0.33031	-5E-06	0.005516	0.27579	-2.7E-06	
0.03	9.9E-05	0.00991	0.33031	-2E-05	0.008274	0.27579	-8.2E-06	
0.04	0.0002	0.01321	0.33031	-3E-05	0.011032	0.27579	-1.6E-05	

Figure 9: Results for Pr=0.6

### CONCLUDING REMARKS

A simple approach for solving the similarity transformed boundary layer equations based on Microsoft's Excel software is presented. The results based on this approach are close to those presented in standard textbooks (within 1% relative error). Although the example presented is for the solution of the Blasius equation (an equation that is commonly covered in undergraduate Fluid Mechanics and Heat Transfer Courses) the present approach can be extended for solving a wide range of problems, e.g., Wedge flow problem and natural convection on a vertical plate.

The author has used this method in his undergraduate heat transfer and graduate convection heat transfer courses. The feedback from students was positive. What the students liked most was the fact that they were able to derive the convective heat transfer correlations. Use of this simple approach removes the need to use complex analytical and numerical approaches. Additionally, since all engineering students are familiar with at least one



spreadsheet software package, it is very easy to incorporate it in the aforementioned courses.

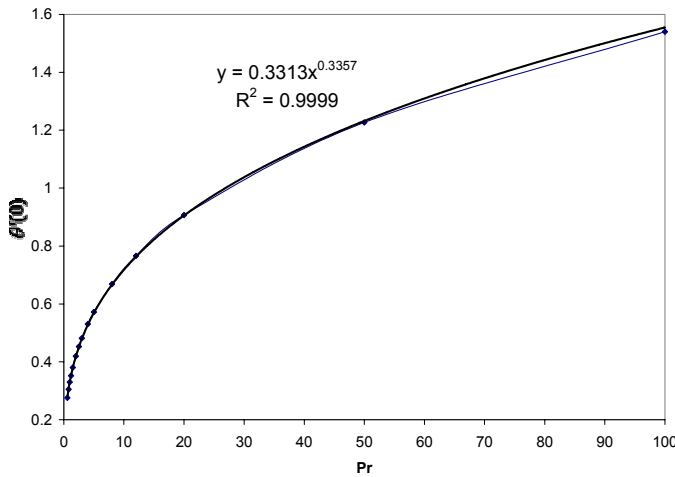


Figure 10:  $\theta'(0)$  versus Prandtl number with trend line equation.

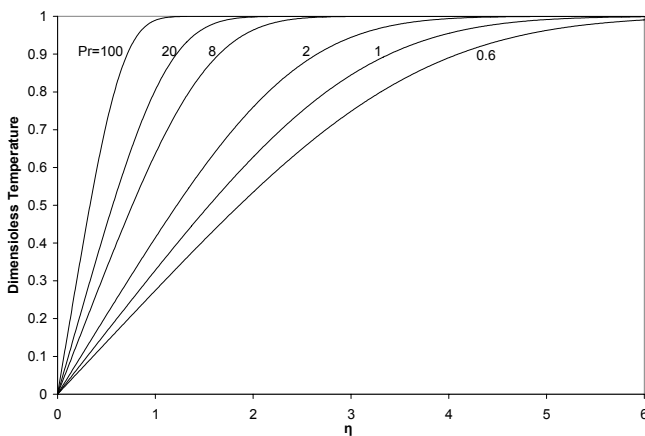


Figure 11: Dimensionless temperature distributions for different Prandtl numbers.

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### BIOGRAPHICAL INFORMATION

Dr. Mohammad Naraghi is a Professor of Mechanical Engineering at Manhattan College. Prior to joining Manhattan College, he was a Visiting Assistant Professor of Mechanical Engineering at the University of Akron where he received his Ph.D. in Mechanical Engineering. Dr. Naraghi worked closely with NASA Glenn Research Center, through research grants and a number of Summer Faculty Fellowships, to develop a comprehensive Rocket Thermal Evaluation code (RTE). Because of this code, he received a certificate of recognition from NASA for the creative development of technically significant software which has been accepted and approved for dissemination to the public by NASA. Since the first release of RTE through NASA's COSMIC library (July 1991), Dr. Naraghi's research is in Thermal/Fluids area and he has published more than sixty articles in ASME, AIAA and international journals and conferences. He is recipient of a number of research grants from NASA and the Air Force. Dr. Naraghi is a Fellow of ASME and a member of AIAA's Liquid Propulsion Technical Committee.

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